Effect of a micro-structure on the thermal striping damage of an edge-cracked slab

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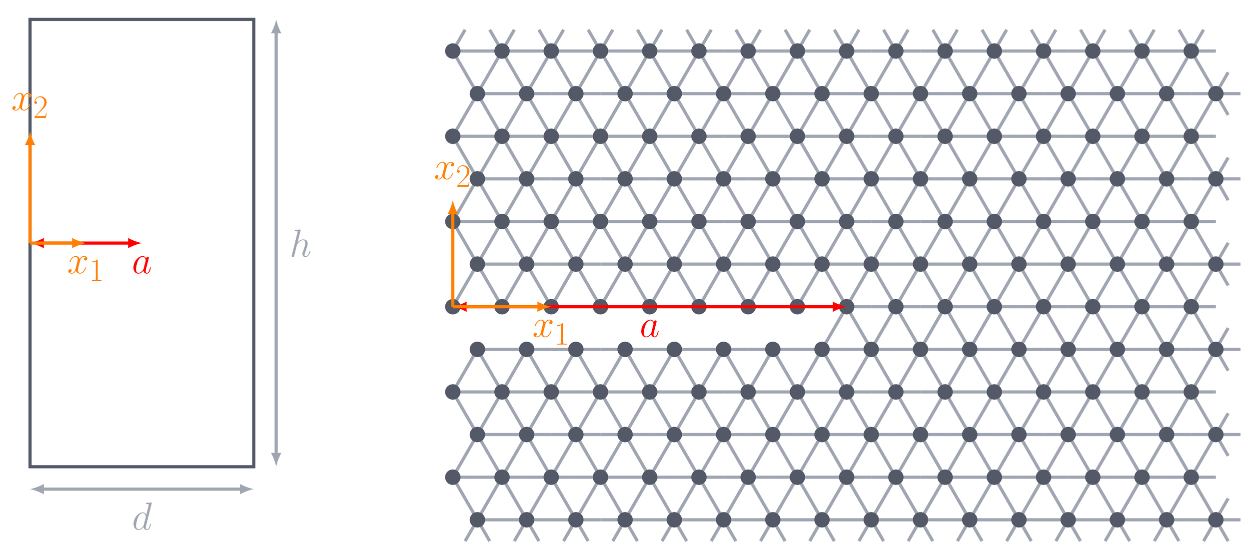
Abstract

In the work reported here, we seek to answer the following question: *Can damage be reduced by developing materials with micro-structure?*. We consider a micro-structured elastic slab containing an edge crack, subjected to sinusoidal thermal striping. In this case, the micro-structure is fully discrete and takes the form of a genuine triangular lattice in the vector setting of planar elasticity. It is well known that , in the continuum, the stress field is singular at the crack tip, whereas for a crack in a lattice, there is merely a stress concentration at the crack front bond. Here, we analyse the local properties of the crack in the lattice and compare these with those of a corresponding homogenised medium. In particular, we introduce the notion of an "effective stress intensity factor" for the edge crack in the lattice, obtained from the crack tip displacements. The effect of varying the number of lattice cells per unit area on this "effective stress intensity factor" is examined and compared with the continuum limit.

The geometry



We consider a finite block of width d and height h, with a finite edge crack of length a located along $x_2 = 0$ as shown on the left. We also consider a corresponding triangular lattice, a section of which is shown on the right. For the work presented here, we consider two lattices with differing degrees of refinement. The first is the *sparse lattice* with links of length $\ell = 2 \times 10^{-4}$m and cross-sectional area $S = 2 \times 10^{-8}\text{m}^2$. The second is referred to as the *fine lattice* with links of length $\ell = 1 \times 10^{-4}$m and cross-sectional area $S = 1 \times 10^{-8}\text{m}^2$.

The uncoupled thermoelastic problem in the continuum

The thermal striping problem in the continuum for the rectangle $\Omega = \{\boldsymbol{x} : 0 < x_1 < d,\ |x_2| < h/2\}$, which contains a finite edge crack $M_a = \{\boldsymbol{x} : 0 \le x_1 \le a,\ x_2 = 0\}$, with the crack faces $M_a^{\pm}$, satisfies the following problem for the elastic displacement field $\boldsymbol{U}(\boldsymbol{x};t)$:

$$\begin{aligned} \mathcal{L}\boldsymbol{U}(\boldsymbol{x};t) &= \alpha(3\lambda + 2\mu)\nabla T(\boldsymbol{x};t), && \boldsymbol{x} \in \Omega \setminus M_a, \\ \boldsymbol{\sigma}^{(\boldsymbol{n})}[\boldsymbol{U}](\boldsymbol{x};t) &= \alpha(3\lambda + 2\mu)\boldsymbol{n}T(\boldsymbol{x};t), && \boldsymbol{x} \in B_0 \cup B_d \cup M_a^+ \cup M_a^-, \\ \boldsymbol{U}(\boldsymbol{x};t) &= \boldsymbol{0}, && \boldsymbol{x} \in \{\boldsymbol{x} : 0 < x_1 < d, |x_2| = h/2\}, \end{aligned} \quad (1)$$

- The Lamé operator: $\mathcal{L} = \mu\Delta + (\lambda + \mu)\nabla[\nabla\cdot]$
- λ and μ are the Lamé coefficients
- α is the coefficient of linear thermal expansion
- $B_r = \{\boldsymbol{x} : x_1 = r,\ |x_2| < h/2\}$
- Traction operator: $\boldsymbol{\sigma}^{(\boldsymbol{n})}[\boldsymbol{U}] = \{\lambda(\nabla\cdot\boldsymbol{U})\mathbb{I} + \mu\{\nabla\boldsymbol{U} + (\nabla\boldsymbol{U})^{\mathrm{T}}\}\}\boldsymbol{n}$
- $\boldsymbol{n}$ and $\mathbb{I}$ are the outward unit normal and 2×2 identity matrix respectively.

The uncoupled thermoelastic problem in the discrete lattice

Consider a uniform triangular meshing of $\mathbb{R}^2$ with nodes at discrete positions $\boldsymbol{x}(\boldsymbol{p}) = \ell\mathfrak{N}\boldsymbol{p}$, where $\boldsymbol{p} \in \mathbb{Z}^2$ labels the nodes separated by distance ℓ and

$$\mathfrak{N} = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}. \quad (2)$$

It is convenient to introduce the following sets of nodes

- Interior nodes: $\Gamma = \{\boldsymbol{p} : 0 < x_1(\boldsymbol{p}) < d,\ |x_2(\boldsymbol{p})| < h/2\}$
- Lateral boundaries: $\gamma_0 = \{\boldsymbol{p} : 0 \le x_1(\boldsymbol{p}) \le \ell/2, |x_2(\boldsymbol{p})| \le h/2\}$ $\gamma_d = \left\{\boldsymbol{p} : d - \frac{\ell}{2} \le x_1(\boldsymbol{p}) \le d, |x_2(\boldsymbol{p})| \le h/2\right\}$
- Horizontal boundaries $\gamma_h = \{\boldsymbol{p} : \ell/2 < x_1(\boldsymbol{p}) < d - \ell/2, |x_2(\boldsymbol{p})| = h/2\}$
- Crack face nodes: $M_a^L = \{\boldsymbol{p} : 0 \le x_1(\boldsymbol{p}) \le a,\ -\sqrt{3}\ell/2 \le x_2(\boldsymbol{p}) \le 0\}$
- Nodes connected to $\boldsymbol{p}$: $\mathcal{C}(\boldsymbol{p}) = \{\boldsymbol{q} : |\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{q}) - \boldsymbol{x}(\boldsymbol{p})| = \ell\} \setminus M_a^L$

The problem for the in-plane elastic displacement $\boldsymbol{v}(\boldsymbol{p})$ of a thermally striped lattice with a finite edge crack is then

$$\begin{aligned} \sum_{\boldsymbol{q}\in\mathcal{C}(\boldsymbol{p})} B(\boldsymbol{q})\{\boldsymbol{u}(\boldsymbol{p}+\boldsymbol{q};t) - \boldsymbol{u}(\boldsymbol{p};t)\} &= \frac{\alpha\ell}{2}\sum_{\boldsymbol{q}\in\mathcal{C}(\boldsymbol{p})} b(\boldsymbol{q})\{\Theta(\boldsymbol{p}+\boldsymbol{q};t) + \Theta(\boldsymbol{p};t)\}, && \boldsymbol{p} \in \Gamma, \\ \boldsymbol{u}(\boldsymbol{p};t) &= \boldsymbol{0}, && \boldsymbol{p} \in \gamma_h, \end{aligned} \quad (3)$$

where $\Theta(\boldsymbol{p};t)$ is temperature at node $\boldsymbol{p}$ at time t. The matrices $B(\boldsymbol{q})$ and vectors $b(\boldsymbol{q})$ describe the direction of the bond linking lattice nodes $\boldsymbol{p}+\boldsymbol{q}$ and $\boldsymbol{p}$:

$$B(\boldsymbol{q}) = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}, \quad b(\boldsymbol{q}) = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad (4)$$

where ϕ is the angle between the point $\mathfrak{N}\boldsymbol{q}$ and the positive x_1-axis.

The heat conduction problem on the continuum

The continuum amplitude ($\theta(\boldsymbol{x}) = T(\boldsymbol{x},t)e^{-i\omega t}$) satisfies the following problem on the rectangle $\Omega = \{\boldsymbol{x} : 0 < x_1 < d,\ |x_2| < h/2\}$

$$\begin{aligned} \kappa\Delta\theta(\boldsymbol{x}) &= i\omega\theta(\boldsymbol{x}), && \boldsymbol{x} \in \Omega, \\ \theta(\boldsymbol{x}) &= T_0, && \boldsymbol{x} \in \Omega \cap \{\boldsymbol{x} : x_1 = 0\}, \\ \theta(\boldsymbol{x}) &= 0, && \boldsymbol{x} \in \Omega \cap \{\boldsymbol{x} : x_1 = d\}, \\ \nabla[\theta(\boldsymbol{x})]\cdot\boldsymbol{p} &= 0, && \boldsymbol{x} \in \Omega \cap \{\boldsymbol{x} : |x_2| = h/2\}, \end{aligned} \quad (5)$$

where ω is the radian frequency of the thermal load and κ is the thermal diffusivity of Ω. Physically (5) corresponds to the time-harmonic thermal striping of a finite conducting rectangle by a sinusoidal load on the left face.

The heat conduction problem on the lattice

The time-harmonic heat conduction problem on a finite lattice may be written in terms of the discrete complex amplitude $\vartheta(\boldsymbol{p})$

$$\begin{aligned} \vartheta(\boldsymbol{p}) &= \frac{1}{i\omega\Xi + |\mathcal{N}(\boldsymbol{p})|}\sum_{\boldsymbol{q}\in\mathcal{N}(\boldsymbol{p})}\vartheta(\boldsymbol{p}+\boldsymbol{q}), && \boldsymbol{p} \in \Gamma, \\ \vartheta(\boldsymbol{p}) &= T_0, && \boldsymbol{p} \in \gamma_0, \\ \vartheta(\boldsymbol{p}) &= 0, && \boldsymbol{p} \in \gamma_d, \\ \vartheta(\boldsymbol{p}) &= \frac{1}{|\mathcal{N}(\boldsymbol{p})|}\sum_{\boldsymbol{q}\in\mathcal{N}(\boldsymbol{p})}\vartheta(\boldsymbol{p}+\boldsymbol{q}), && \boldsymbol{p} \in \gamma_h, \end{aligned} \quad (6)$$

here $\Xi = C\ell/(S\lambda)$ and $\mathcal{N}(\boldsymbol{p}) = \{\boldsymbol{q} : |\boldsymbol{x}(\boldsymbol{p}+\boldsymbol{q}) - \boldsymbol{x}(\boldsymbol{p})| = \ell\}$ denotes the set of nodes connected to node $\boldsymbol{p}$, with $\boldsymbol{q} \in \mathbb{Z}^2$.

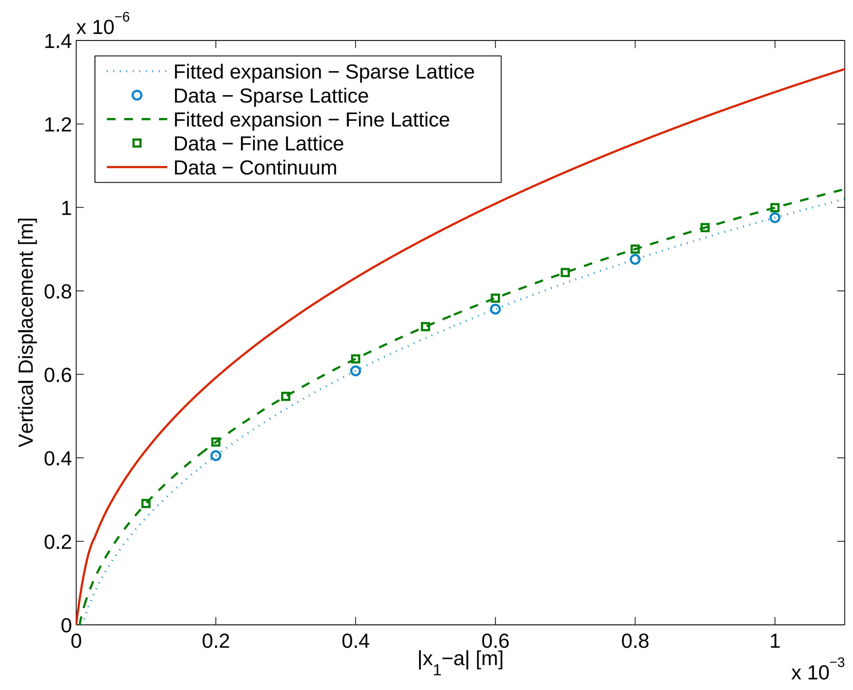
An effective stress intensity factor for the lattice

For a sufficiently refined lattice the vertical displacements behind the crack tip exhibit similar asymptotic behaviour to the continuum. In particular, it is assumed that for a sufficiently refined lattice

$$u_2(\boldsymbol{p}) \sim \frac{K_I}{(1-k^2)\mu}\sqrt{\frac{a - x_1(\boldsymbol{p})}{2\pi}} + b_1[a - x_1(\boldsymbol{p})] + b_2[a - x_1(\boldsymbol{p})]^{3/2} + b_3[a - x_1(\boldsymbol{p})]^2, \quad (7)$$

for $\boldsymbol{p} \in \{\boldsymbol{p} : x_1(\boldsymbol{p}) < a,\ x_2(\boldsymbol{p}) = 0\}$ and where $k = 3 - 4\nu$ and μ is the shear modulus corresponding to the homogenised continuum.

The figure shows that the expansion (7) is sufficient to accurately capture the behaviour of the u_2 displacements behind the crack tip and that the displacements exhibit the same qualitative behaviour as in the continuum.
In direct analogy to the displacement extrapolation method for the continuum, an effective stress intensity factor at a particular time may be determined by fitting the expansion (7) to the displacements behind the crack tip.

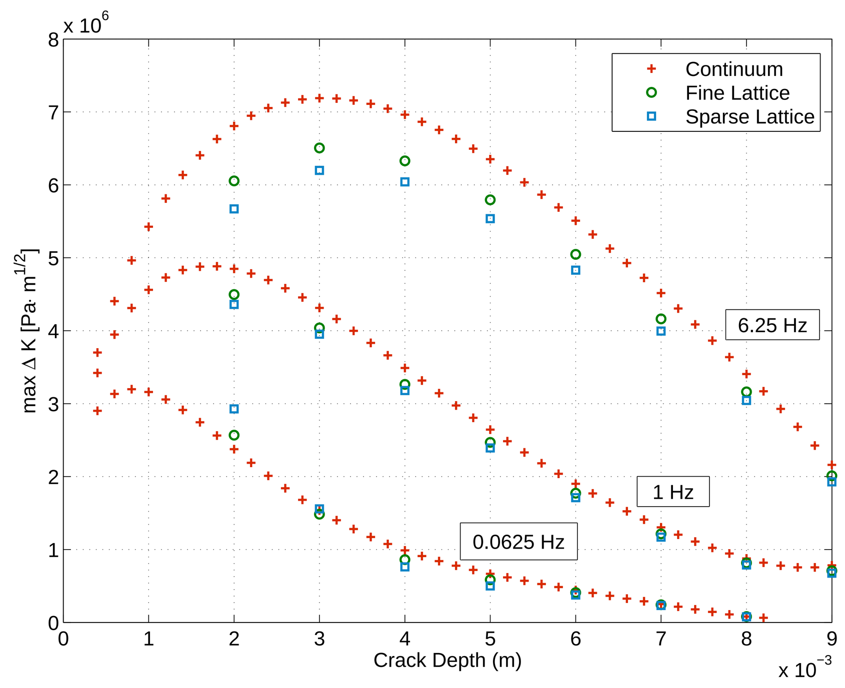


For the continuum the stress intensity factor is evaluated from the finite element results using a variant of the J integral, which for plane strain is related to K_I by

$$\frac{1-\nu^2}{E}K_I^2 = \frac{1}{2}\int_\gamma\left(\sigma_{ij}\epsilon_{ij} - \frac{E\zeta}{1-2\nu}T\epsilon_{ii}\right)\mathrm{d}x_2 - \int_\gamma \sigma_{ij}n_j\frac{\partial U_i}{\partial x_1}\mathrm{d}s + \frac{E\zeta}{1-2\nu}\int_\Gamma \epsilon_{ii}\frac{\partial T}{\partial x_1}\mathrm{d}A, \quad (8)$$

where the arbitrary curve γ encloses the area Γ, σ_{ij} is the usual thermoelastic stress tensor, ϵ_{ij} are the components of strain, E is Young's modulus. The line and area integrals in (8) were computed from the finite element results using fourth order quadrature over three contours in the vicinity of the crack tip.

The stress intensity factor for the lattice vs. continuum



The figure shows the maximum [effective] ΔK_I values for the thermally striped continuum and the two lattices at three striping frequencies: 0.0625Hz, 1Hz and 6.25Hz. The continuum curves show similar behaviour to that observed in earlier works, with the local maxima of ΔK_I increasing and shifting further to the right for lower frequencies. For sufficiently long cracks, the lattice curves exhibit the same qualitative behaviour as the continuum.

Compared with the continuum, the lattices have a reduced effective stress intensity factor, except for shorter cracks at higher frequencies. It is also apparent that the more "refined" the lattice, the closer the stress intensity factor is to the continuum value. For shorter edge cracks (smaller than 2×10^{-3}m) the nodal displacements no longer exhibit the square root asymptotic behaviour.

Parametric values

Parameter		
Symbol	Description	Numerical Value
S/ℓ	Ratio of the length of the lattice links to cross-sectional area (m)	10^{-4}
T_0	Amplitude of thermal striping load (°C)	10
κ	Thermal diffusivity (m^2/s)	2.29×10^{-5}
h	Block height (m)	$1.16\sqrt{3} \times 10^{-2}$
d	Block width (m)	10^{-2}
E	Young's Modulus (GPa)	163.5
ν	Poisson's ratio	$1/4$
α	Linear thermal expansion coefficient (1/°C)	2×10^{-5}

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