# Structured interfaces for flexural waves - trapped modes and transmission resonances

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**Abstract.** The article combines the analytical models of scattering and Bloch waves for a stack of periodic gratings in an infinite elastic plate. The waves represent flexural deflections of the plate governed by a fourth-order partial differential equation. The emphasis is on the analysis of trapped modes and transmission resonances for different configurations of the grating stack and physical parameters of the flexural waves. Special attention is given to the phenomenon of Elasto-Dynamically Inhibited Transmission (EDIT). The analytical model is supplemented with comprehensive numerical examples.

## 1. Introduction

An important branch of structural mechanics is the study of the vibration of plates. The analysis of such models gives us an insight into the predicted response of elastic structures to earthquakes and other dynamic loads. In this paper we consider structured plates that contain an interface consisting of a finite number of periodic gratings, and our particular interest is in the localisation of flexural waves within the grating structure. Many elastic systems such as aircraft, long bridges and reinforced roads encompass periodic arrangements of inclusions, voids and masses. These structures are frequently subjected to stress concentrations and therefore the study of the resonant action of incident flexural waves within our structured plates has applications in engineering.

Our model allows for the gratings to consist of inclusions of any shape or size. Here we will concentrate on rigid, cylindrical voids with a finite radius, in the limiting case of the radius tending to zero, which can be thought of as giving a pinned point. We demonstrate that the periodically structured interface supports sharp transmission resonances for low-frequency flexural vibrations arising from the interaction of a plane wave, characterised by its angle of incidence $\theta_i$ and spectral parameter $\beta$, with the plate.

We show an illustrative example in Fig. 1. The outer pair of this triplet consists of pinned points, whilst the central grating contains rigid voids with a finite radius $a$. The other parameters are the angle of incidence $\theta_i$, the period of the gratings $d$ and the grating separation $\eta$. We also indicate the scattered field. An important additional parameter is the relative shift of the central grating denoted by $\xi$, which is crucial for supporting a filtering effect similar to Electromagnetically Induced Transparency (EIT). We term this Elasto-Dynamically Inhibited

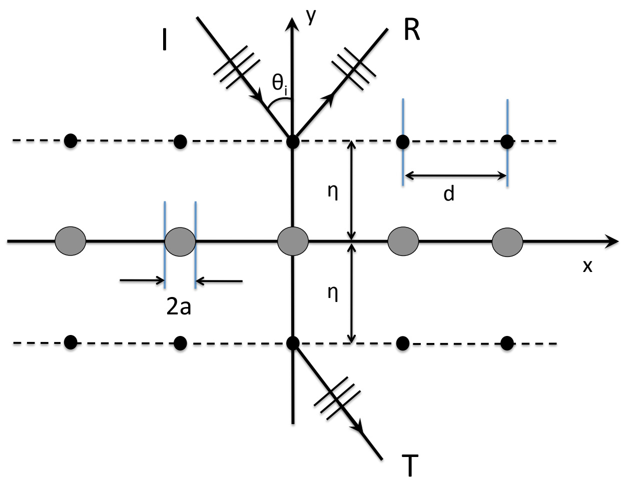


**Figure 1.** Stack of gratings consisting of an outer pair of rigid pins and a central grating of finite nonzero inclusions with radius $a$ and period $d$. The relative grating separation between consecutive gratings is denoted by $\eta$.

Transmission (EDIT) and this novel phenomenon for elasticity problems was first observed by Haslinger*et al.* [1-2]. It is characterised by a resonant peak in transmission being cut in two by a resonant dip with an extremely high quality factor.

We discuss how the radius of the inclusions affects EDIT, which is dependent on the coincidence of symmetric and anti-symmetric modes. For a triplet consisting only of rigid pins, the anti-symmetric mode's frequency is invariant with $\xi$ [1]. This makes it easier to tune the system's symmetric mode to coincide with the odd mode. This is no longer true for inclusions of nonzero radius.

The increased radius makes it necessary to take into account higher-order multipole terms characterising the scattered field, and the periodicity of the grating will also lead to the use of higher order lattice sums. It follows that more care is required to locate the frequency for EDIT for structures incorporating finite-sized voids.

Furthermore, the grating stack can be considered as a structured waveguide for trapped modes. For a simple case of a stack of rigid pins, a quasi-periodic Green's function is employed to derive the dispersion equation for Bloch waves in such a waveguide. A connection is established with the transmission problem by identifying parameters of the grating stack and of the incident wave, to generate a transmission resonance linked to a trapped Bloch wave within the structured waveguide.

## 2. Transmission resonances for gratings stacks

A structured interface consisting of a periodic array of cells of chosen geometry separates two half-planes in an infinite flexural plate. A plane wave representing flexural displacements is incident on such an interface, and it is shown that a nearly 100 % transmission is achievable for a certain combination of parameters within a narrow frequency range. A new phenomenon of Elasto-Dynamically Inhibited Transmission is outlined for a stack of three gratings.

*2.1. Governing equations and method of solution*

Let the amplitude $W$ be a solution of the scattering problem for the biharmonic operator (see Movchan *et al.* [3]):

$$\Delta^2 W(\mathbf{x}) - \beta^4 W(\mathbf{x}) = 0, \tag{1}$$

where $\beta^2 = \omega\sqrt{\rho h/D}$. Within this expression, $D = Eh^3/(12(1-\nu^2))$ is the flexural rigidity of the plate, $h$ denotes the plate thickness, $E$ is the Young modulus and $\nu$ is the Poisson ratio. In addition, $\rho$ is the mass density and $\omega$ is the angular frequency. Here we have assumed time-harmonic vibrations with angular frequency $\omega$. Consequently, equation (1) is derived by substituting the out-of-plane displacement $w(\mathbf{x};t) = W(\mathbf{x})\sin(\omega t)$ into the standard Kirchhoff plate equation of motion:

$$D\Delta^2 w(\mathbf{x};t) \, + \, \rho h \frac{\partial^2 \omega}{\partial t^2}(\mathbf{x};t) \, = \, 0. \tag{2}$$

This yields an equation satisfied by the amplitude $W(x,y)$ or $W(r,\theta)$, where $(r,\theta)$ are the polar coordinates applicable to the case of cylindrical inclusions.

The solution of equation (1) can be divided into two parts $W_H$ and $W_M$, which satisfy the Helmholtz equation and its counterpart form, the modified Helmholtz equation:

$$(\Delta + \beta^2)W_H = 0, \qquad (\Delta - \beta^2)W_M = 0. \tag{3}$$

An important aspect of the physics of the problem is that $W_H$ contains both propagating and evanescent waves, while $W_M$ consists entirely of evanescent waves.

The field $W$ satisfies the Bloch quasi-periodicity condition along the horizontal $x$-axis:

$$W(\mathbf{x} + md\mathbf{e}^{(1)}) = W(\mathbf{x})e^{i\alpha_0 md}, \tag{4}$$

where the diffraction order $m$ is an integer, $d$ is the period, and $\alpha_0$ is the Bloch parameter $\alpha_0 = \beta\sin\theta_i$, where $\theta_i$ is the angle of incidence (see Fig. 1).

The boundary conditions on the circular boundaries of the inclusions are Dirichlet clamping conditions:

$$W\bigg|_{r=a} = 0, \quad \frac{\partial W}{\partial r}\bigg|_{r=a} = 0. \tag{5}$$

We initially consider the limit as $a \to 0$, corresponding to an array of rigid pins constraining the plate. The radius of the inclusions is important for the implementation of the method that we use to solve the problem, as outlined in the papers by Movchan *et al.* [3-6] and Haslinger *et al.* [1-2].

We consider the scattering of plane waves, either of the Helmholtz type $W_H$ or of the modified Helmholtz type $W_M$, by a grating of inclusions of radius $a$. The flexural displacement $W$ is expanded for $y > a$ and $y < -a$ in terms of plane waves of type $W_H$ and $W_M$. Above the grating the expansion has a down-going incident wave term and up-going reflected waves with amplitude $R_p$ and $\hat{R}_p$, for the respective wave types. Below the grating the amplitudes of the down-going transmitted waves are denoted by $T_p$ and $\hat{T}_p$. In order to connect these two types of expansions we introduce multipole expressions for $W$ in the region $-a \le y \le a$. The multipole expansion for $W_H$ involves cylindrical waves $J_n(\beta r)e^{in\theta}$ and $H_n^{(1)}(\beta r)e^{in\theta}$ with respective amplitudes $A_n$ and $E_n$:

$$W_H(\mathbf{x}) = \sum_{n=-\infty}^{\infty} \{A_n J_n(\beta r) + E_n H_n^{(1)}(\beta r)\}e^{in\theta}. \tag{6}$$

The multipole expansion for $W_M$ involves modified Bessel function terms $I_n(\beta r)e^{in\theta}$ and $K_n(\beta r)e^{in\theta}$ with respective amplitudes $B_n$ and $F_n$:

$$W_M(\mathbf{x}) = \sum_{n=-\infty}^{\infty} \{B_n I_n(\beta r) + F_n K_n(\beta r)\}e^{in\theta}, \tag{7}$$

The amplitudes $A_n, B_n, E_n, F_n$ are the multipole coefficients to be determined, and they are related by the boundary conditions (5).

The Rayleigh identity expresses the part of the expansion for $W$ which is regular near the origin (i.e. the terms involving the Bessel functions $J_n$ and $I_n$) as sums over the part which is irregular near the origin (i.e. the terms involving the Bessel functions $H_n^{(1)}$ and $K_n$), together with a term representing the expansion of the incident wave in multipoles. The resulting equations are uncoupled for solutions of the Helmholtz and the modified Helmholtz type. The necessary coupling between the two types of waves is provided by the boundary conditions.

The combination of the Rayleigh identity and the boundary conditions gives a system of linear equations, which is truncated (e.g. so that Bessel functions of orders $-L$ to $L$ are retained) and solved to evaluate a set of multipole coefficients. These in turn are used to evaluate plane wave amplitude coefficients for the fields above and below the grating, using reconstruction equations (see Movchan *et al* [3-4]). The amplitudes of the reflected and transmitted waves are determined for the set of incident fields corresponding to a range of grating orders $p$, both propagating and evanescent. These are assembled into scattering matrices for reflection and transmission that completely characterise the grating's scattering action, and enable us to identify transmission resonances for a finite number of gratings.

*2.2. Transmission resonances and Elasto-Dynamically Inhibited Transmission*

The formulae for transmission and reflection matrices have already been obtained for a single grating, together with the formulae for the transmission and reflection matrices for a pair of identical gratings (see Movchan *et al.* [5])

$$\begin{aligned} \boldsymbol{\mathcal{T}}_2 &= \boldsymbol{\mathcal{T}}_1[\mathbf{I} - (\boldsymbol{\mathcal{R}}_1)^2]^{-1}\boldsymbol{\mathcal{T}}_1, \\ \boldsymbol{\mathcal{R}}_2 &= \boldsymbol{\mathcal{R}}_1 + \boldsymbol{\mathcal{T}}_1\boldsymbol{\mathcal{R}}_1[\mathbf{I} - (\boldsymbol{\mathcal{R}}_1)^2]^{-1}\boldsymbol{\mathcal{T}}_1. \end{aligned} \tag{8}$$

The matrices $\boldsymbol{\mathcal{T}}_i$, $\boldsymbol{\mathcal{R}}_i$ comprise the matrices $\mathbf{T}_i$ and $\mathbf{R}_i$ respectively, where $i = 1, 2$, which are formed using the coefficients for the plane wave representations, together with a diagonal propagation matrix $\boldsymbol{\mathcal{P}}$:

$$\boldsymbol{\mathcal{R}}_i = \boldsymbol{\mathcal{P}}\mathbf{R}_i\boldsymbol{\mathcal{P}}, \ \boldsymbol{\mathcal{T}}_i = \boldsymbol{\mathcal{P}}\mathbf{T}_i\boldsymbol{\mathcal{P}}. \tag{9}$$

The propagation matrix incorporates the effect of both vertical separation of the gratings $\eta$ and relative lateral shift $\xi$:

$$\boldsymbol{\mathcal{P}} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}, \qquad \text{where} \qquad P = [\delta_{tp} e^{i(\tilde{\chi}_p \eta/2 - \alpha_p \xi/2)}], \tag{10}$$

with $\tilde{\chi}_p = \chi_p$ if $p$ corresponds to a Helmholtz type plane wave and $\tilde{\chi}_p = \hat{\chi}_p$ if $p$ corresponds to a plane wave of modified Helmholtz type.

The idea of transmission resonances for gratings of rigid pins was addressed by Movchan *et al.* [4] and Haslinger *et al.* [1-2]. The increase of the radius of inclusions within the gratings affects the scattered fields, with a change in the frequency of the resonance mode as well as its quality factor $Q$. For a resonant peak of transmittance $T$ occurring at $\beta = \beta_*$ with $T = T_{\max}$ there, if $T = T_{\max}/2$ for $\beta = \beta_+$ and $\beta_-$, then $Q = \beta_*/|\beta_+ - \beta_-|$.

We demonstrate the EDIT phenomenon with a triplet of gratings which typically supports two transmission resonances, an even and an odd mode. Haslinger *et al.* [2] described how the frequency of the odd mode is invariant with the relative lateral shift of the central grating, and this factor ensures that the EDIT shift $\xi$ is simpler to tune by moving the symmetric mode to coincide with the anti-symmetric one. Illustrative examples for gratings of rigid pins are discussed by Haslinger *et al.* [1-2]. The latter paper also includes the case whereby the outer

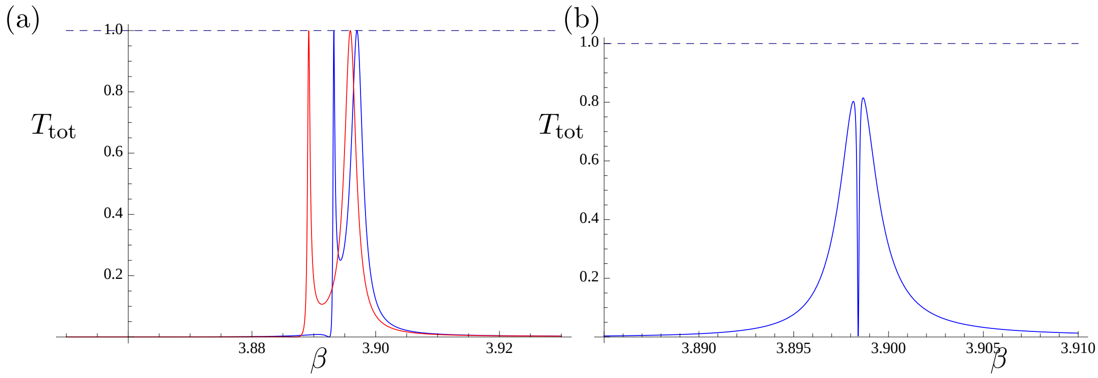


**Figure 2.** Total transmittance $T_{\rm tot}$ as a function of $\beta$ for a triplet with a shifted central grating of finite radius inclusions with $a = 0.0035d$, and an outer pair of gratings of inclusions of radius $a = 0.01d$, $\theta_i = 26°$. Data used: (a) $\xi = 0.4d$ (red curve), $\xi = 0.42d$ (blue curve). (b) $\xi = 0.45556d$.

pair of rigid pins is replaced by inclusions of finite radius, and the central grating of rigid pins is shifted to facilitate the EDIT effect.

In Fig. 2(a), we consider a triplet with an outer pair consisting of inclusions of radius $a = 0.01d$, and a central grating with $a = 0.0035d$. Two shifts of the central grating are illustrated, $\xi = 0.4d$ shown by the red curve, and $\xi = 0.42d$ in blue. It is clear that there is no invariant mode, but the odd mode has a higher value of $\beta$ and is being perturbed more slowly than the even mode, which is brought closer to its partner by an increase in $\xi$. By tuning this value of $\xi$ carefully, we show the characteristic dip to zero transmittance of the EDIT phenomenon for the alignment of the two modes at $\xi = 0.45556d$ in Fig. 2(b).

## 3. Waveguides and quasi-periodic grating Green's function

We design a waveguide consisting of a triplet of rigid pin gratings. There are similarities with the scattering problem that we have previously analysed. In that case we identified extremely narrow frequency bands that supported transmission resonances for an incident plane wave characterised by its angle of incidence. The combination of the specific angle of incidence $\theta_i$ and its corresponding spectral parameter $\beta$ means that each transmission peak is defined by a specific value of the Bloch parameter $\alpha_0 = \beta \sin\theta_i$. Therefore we pay particular attention to these Bloch values for the waveguide problem, with the expectation that they will support localisation within the grating structure.

### *3.1. Quasi-periodic grating Green's function and dispersion equation*

We may consider a single grating of rigid pins as a line of point forces with constant separation $d$. Therefore we use a quasi-periodic Green's function of the form

$$G(\mathbf{r};\beta,\alpha_0) \;=\; \sum_{n=-\infty}^{\infty} g(\mathbf{r} - nd\mathbf{e^{(1)}},\beta)\, e^{i\alpha_0 nd} \tag{11}$$

to describe a single grating. Evans and Porter [7] give the fundamental Green's function for a time harmonic point source:

$$g\,(\mathbf{r};\beta) \;=\; -\frac{i}{8\beta^2}\left(H_0^{(1)}(\beta r) \;-\; H_0^{(1)}(i\beta r)\right), \tag{12}$$

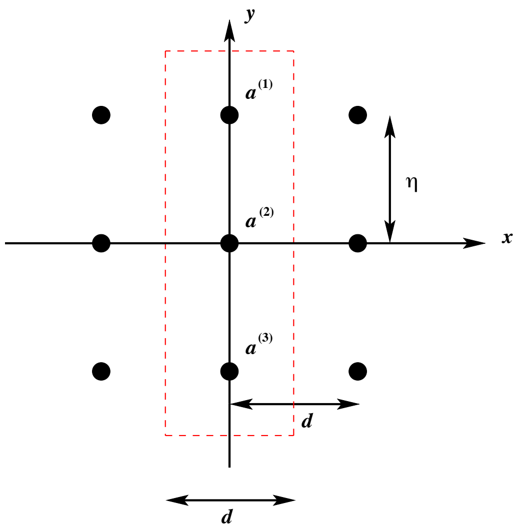


**Figure 3.** Elementary cell for the waveguide consisting of a triplet of unshifted rigid pin gratings.

which solves the equation

$$(\Delta^2 - \beta^4) g(\mathbf{r}; \beta) + \delta(\mathbf{r}) \;=\; 0. \tag{13}$$

The corresponding quasi-periodic Green's function is finite at the origin, and can be written as follows

$$G(x, y; \beta, \alpha_0) = \frac{1}{4d\beta^2}\Big(i \sum_{n=-\infty}^{\infty} \chi_n^{-1} \exp(i[\alpha_n x + \chi_n |y|]) - \sum_{n=-\infty}^{\infty} \tau_n^{-1} \exp(i\alpha_n x - \tau_n |y|)\Big), \tag{14}$$

where

$$\alpha_n = \alpha_0 + \frac{2\pi n}{d},$$

$$\chi_n = \begin{cases} \sqrt{\beta^2 - \alpha_n^2}, & \alpha_n^2 \le \beta^2 \\ i\sqrt{\alpha_n^2 - \beta^2}, & \alpha_n^2 > \beta^2, \end{cases}$$

$$\tau_n = \sqrt{\beta^2 + \alpha_n^2}.$$

For more details of their evaluation, see Movchan *et al.* [4-5], Haslinger *et al.* [1-2].

For convenience, we introduce grating Green's functions $G^{(j)}$ in such a way that

$$G^{(1)}(x, y; \beta, \alpha_0) = G(x, y - \eta; \beta, \alpha_0), \; G^{(2)}(x, y; \beta, \alpha_0) = G(x, y; \beta, \alpha_0),$$

$$G^{(3)}(x, y; \beta, \alpha_0) = G(x, y + \eta; \beta, \alpha_0).$$

For an elementary cell of the form shown in Fig. 3, we construct a matrix of Green's functions $\mathbf{M}$ which fully describes the triplet waveguide, and satisfies the dispersion equation

$$\mathbf{MA} \;=\; 0, \tag{15}$$

where $M_{ij} = G^{(j)}(0, (2 - i)\eta; \beta, \alpha_0), i, j = 1, 2, 3,$ $\mathbf{A}$ is a column vector of coefficients $A_j$ to be determined. For the case of an unshifted triplet, we obtain a symmetric Toeplitz matrix, with

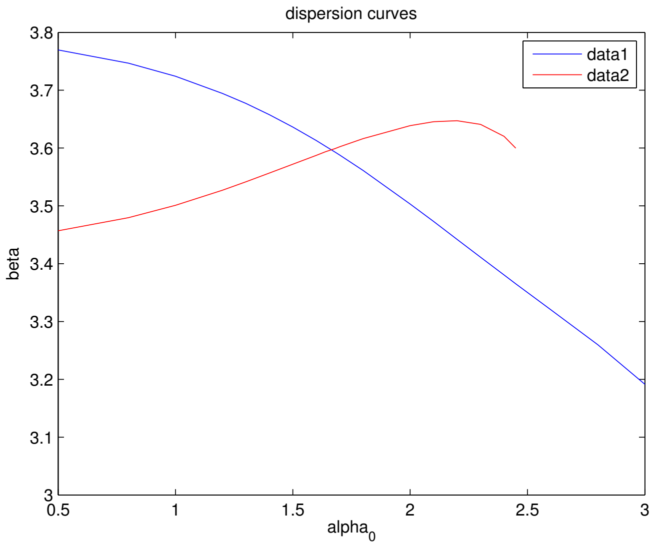


**Figure 4.** Dispersion diagram for a waveguide consisting of an unshifted triplet with the horizontal axis representing $\alpha_0$ in the range $0 \leq \alpha_0 \leq \pi$, and $\beta$ on the vertical axis in the range $2 \leq \beta \leq 4$. Produced using *Matlab*. The red curve represents the odd modes, and the blue curve, the even modes.

complex entries which describe the contribution of each grating layer to the displacement at the location of the pinned point under study:

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{11} & M_{12} \\ M_{13} & M_{12} & M_{11} \end{pmatrix}. \tag{16}$$

The matrix has three eigenvalues:

$$\lambda_1 = M_{11} - M_{13}, \quad \lambda_{\pm} = \frac{1}{2}(2M_{11} + M_{13} \pm \sqrt{8M_{12}^2 + M_{13}^2}). \tag{17}$$

The eigenvector corresponding to $\lambda_1$ is

$$v_o = \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix}, \tag{18}$$

and has odd symmetry. The other two vectors have even symmetry.

Dispersion curves for the modes are the trajectories along which the eigenvalues are zero (or more precisely, since we have complex entries, have a modulus close to zero). For the odd mode, we have the requirement

$$M_{11} - M_{13} = 0 \text{ or } |M_{11} - M_{13}| \simeq 0. \tag{19}$$

For the even modes, the dispersion curves correspond to

$$M_{12}^2 = \frac{1}{2}M_{11}(M_{11} + M_{13}). \tag{20}$$

We are primarily interested in the two lowest branches of the waveguide's dispersion curves, illustrated in Fig. 4.

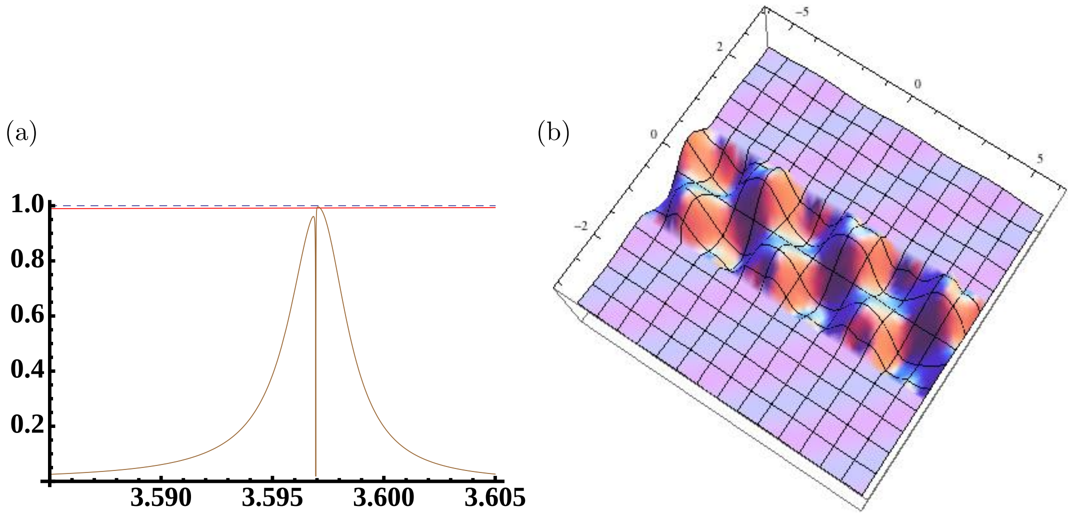


**Figure 5.** Normalised transmitted energy versus spectral parameter $\beta$ for a triplet of rigid pins with angle of incidence $\theta_i = 0.481273$ and shift of the central grating $\xi = 0.012$ (b) Field plots for the example illustrated in (a).

We find that the odd and even modes cross. The even modes are represented by the curve with lower values of $\beta$ after the intersection point, and they satisfy the condition

$$M_{12} = -\sqrt{M_{11}(M_{11} + M_{13})/2}. \tag{21}$$

The crossing of the two dispersion curves is synonymous with a double eigenvalue case, and is of great interest. In Fig. 5, we consider the corresponding transmission resonance example for a specific angle of incidence in the vicinity of the intersection point, and we observe an EDIT-like effect, when we introduce a small non-zero shift of the central grating $\xi = 0.012d$.

## 4. Tuning of the resonance transmission system

The two approaches outlined above present the grating stack as a periodic scatterer and as a waveguide. The results of analysis complement each other in a way that enables us to design a system possessing Elasto-Dynamically Inhibited Transmission (EDIT).

Based on a straightforward analysis of a pair of gratings of rigid pins, one evaluates the transmission coefficients and obtains resonance regimes as shown in Fig. 6. In part (a) we show the transmission resonances for a pair of rigid pin gratings with vertical separation $\eta = d$ (red curve), and $\eta = 2d$ (blue curve). In part (b) we illustrate the corresponding unshifted triplet with the same angle of incidence $\theta_i = 30^\circ$. We note that the anti-symmetric mode with $\beta = 3.61747$ is common to both diagrams. The symmetric mode of the triplet is linked to the mode for $\eta = d$ featured in part (a) with $\beta = 3.58221$.

Furthermore, an addition of a third grating changes the symmetric resonance modes. The required tuning of the grating stack as a scattering system is rather complex, but it becomes a straightforward task with the knowledge of dispersion properties of trapped flexural waves, characterised by the dispersion curves in Fig. 4. The intersection point between these curves corresponds to the frequency in the neighbourhood of EDIT. Further refinement is achieved by the choice of the shift parameter $\xi$. An example for a rigid pin triplet is illustrated in Fig. 7.

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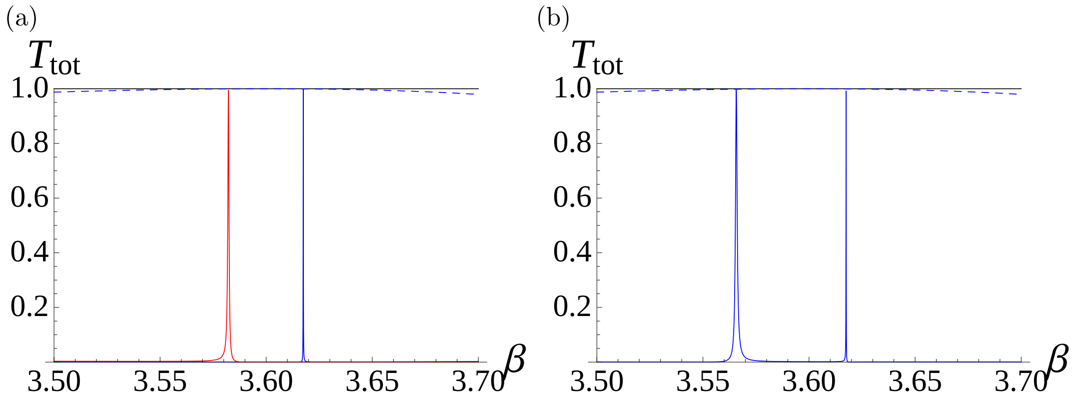


**Figure 6.** Normalised transmitted energy versus spectral parameter $\beta$ for $\theta_i = 30°$ for (a) a pair of rigid pin gratings with $\eta = d$ (red curve) and $\eta = 2d$ (blue curve). (b) a triplet of rigid pin gratings.

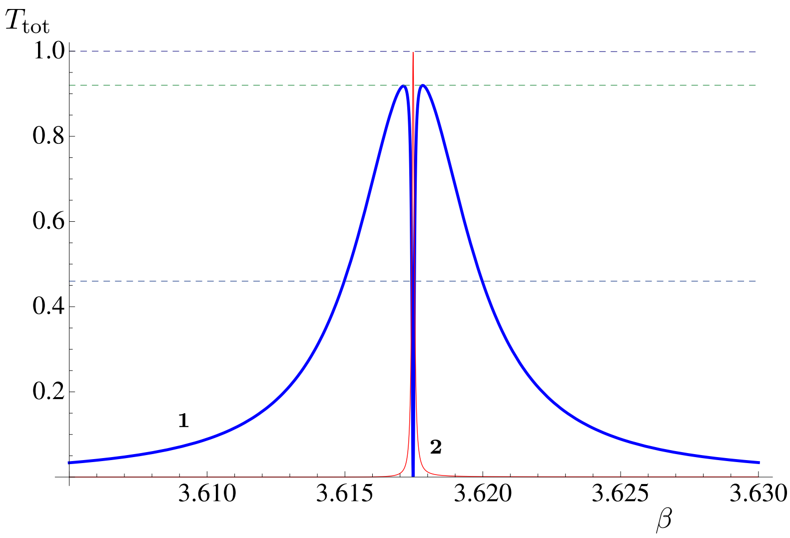


**Figure 7.** EDIT effect for a triplet of rigid pin gratings with the central grating shifted by $\xi = 0.25200d$ for the angle of incidence $\theta_i = 30°$. Total transmittance $T_{\text{tot}}$ (curve **1**) as a function of $\beta$ for the triplet. Curve **2** represents the total transmittance for the outer pair of gratings.

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