Dynamic anisotropy and primitive waveforms in discrete elastic systems

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Outline

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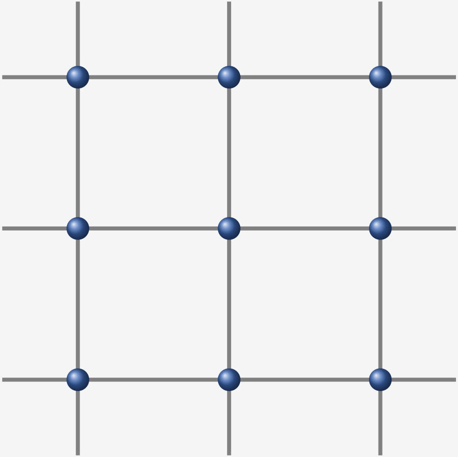
   - Green's function & primitive wave forms
3. Triangular lattices
   - Scalar problem
     - Green's function & primitive waveforms
   - Vector problem - planar elasticity
     - Governing Equations
     - Dispersive properties
     - Primitive Waveforms

Existing Literature

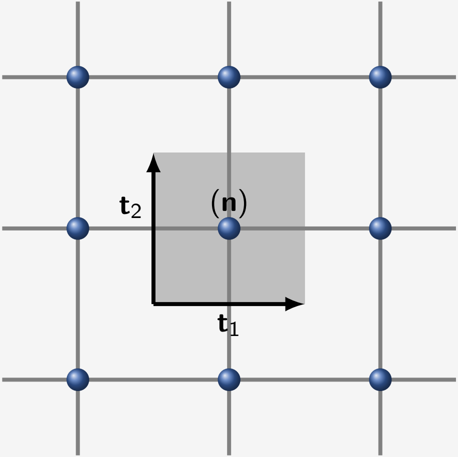
- Dispersion & design of structures for controlling stop bands in concentrated mass systems
  - P. Martinsson & A. Movchan, QJMAM 56 (2003), pp. 45–64.
- Primitive waveforms in scalar lattices
  - Ayzenberg-Stepanenko & Slepyan, J Sound Vib 313 (2008), pp 812–821.
  - Osharovich et al., Continuum Mech Therm 22 (2010), pp. 599–616.
  - Langley, J Sound Vib 197 (1996), pp. 447–469.
  - Langley, J Sound Vib 201 (1997), pp. 235–253.
- Dynamic homogenization of periodic media
  - Craster et al., QJMAM 63 (2010), pp.497–519.
  - Craster et al., Proc R Soc A 466 (2010), pp. 2341–2362.
  - Craster et al., JOSA A (2011), pp. 1032–1040.

A uniform square lattice



- Label each node by $\mathbf{n} \in \mathbb{Z}^2$
- Introduce counting vectors $\mathbf{e}_i = [\delta_{1i}, \delta_{2i}]^T$.
- Lattice vectors: $\mathbf{t}_1 = [\ell, 0]^T$, $\mathbf{t}_2 = [0, \ell]^T$.
- Translation matrix $\mathcal{T} = [\mathbf{t}_1, \mathbf{t}_2]$.
- Position of $\mathbf{n}^{\text{th}}$ node: $\mathbf{x} = \mathbf{n} \otimes \mathcal{T}$.

A uniform square lattice



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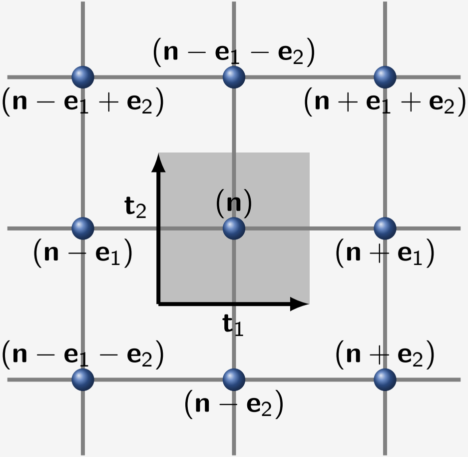
A uniform square lattice



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- Position of $\mathbf{n}^{\text{th}}$ node: $\mathbf{x} = \mathbf{n} \otimes \mathcal{T}$.

Uniform Square Lattice

Green's function for time-haromic out-of-plane shear

Balance of linear momentum at node $\mathbf{n}$ connected to set of nodes $\mathbb{P}(\mathbf{n}) = \{\mathbf{n} \pm \mathbf{e}_1, \mathbf{n} \pm \mathbf{e}_2\}$ (for time-harmonic deformations)

$$\sum_{\mathbf{p}\in\mathbb{P}(\mathbf{n})} u(\mathbf{p}) - |\mathbb{P}(\mathbf{n})|u(\mathbf{n}) + \omega^2 u(\mathbf{n}) = \delta_{m0}\delta_{n0}$$

Discrete Fourier transform:

$$\mathcal{F} : u(\mathbf{m}) \mapsto \mathcal{U}(\boldsymbol{\xi}) = \sum_{\mathbf{m}\in\mathbb{Z}^2} u(\mathbf{m}) \exp\{-i\boldsymbol{\xi}\cdot(\mathbf{n}\otimes\mathcal{T})\}$$

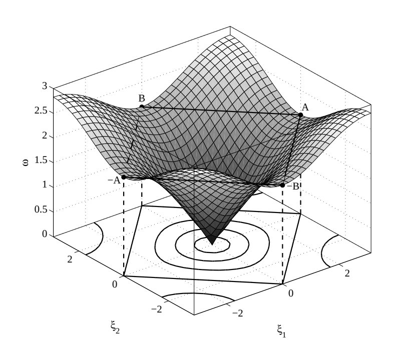
Inverse transform:

Green's Function

$$u(\mathbf{n}) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{\cos(n_1\xi_1)\cos(n_2\xi_2)}{\omega^2 - 4 + 2(\cos\xi_1 + \cos\xi_2)} \mathrm{d}\xi_1 \mathrm{d}\xi_2.$$

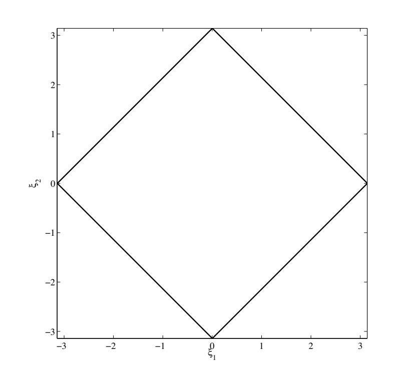
Primitive Wave Forms at a Saddle Point

Dispersion Diagram



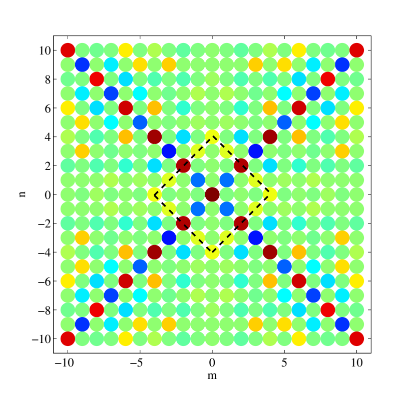
$$\sigma(\omega, \boldsymbol{\xi}) = 0$$

Slowness Contour

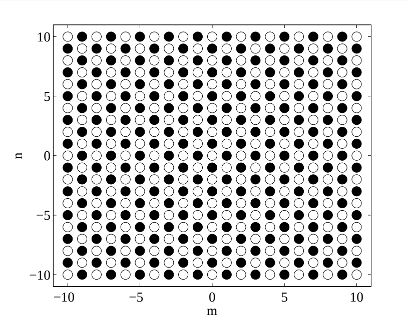


$$\sigma(2, \boldsymbol{\xi}) = 0$$

Two stationary points of different kinds

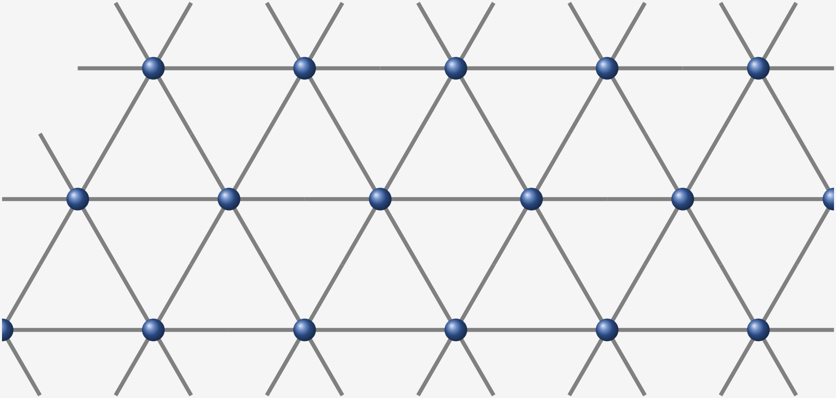


$\omega = 2$



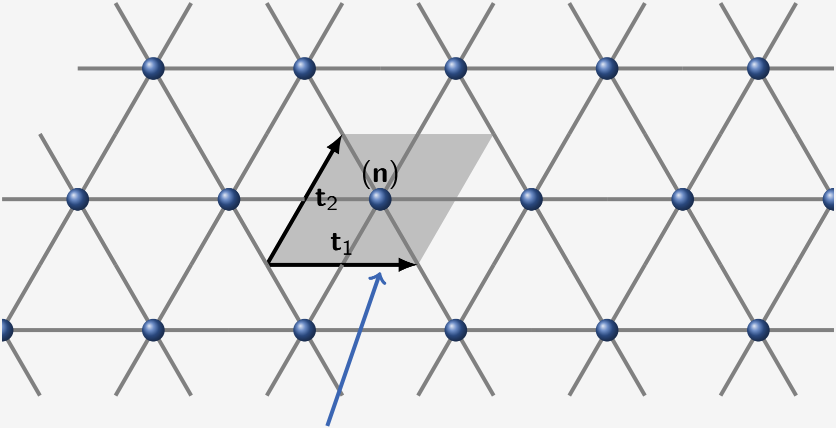
$\omega = 2\sqrt{2}$

Uniform Triangular Lattice



Uniform Triangular Lattice

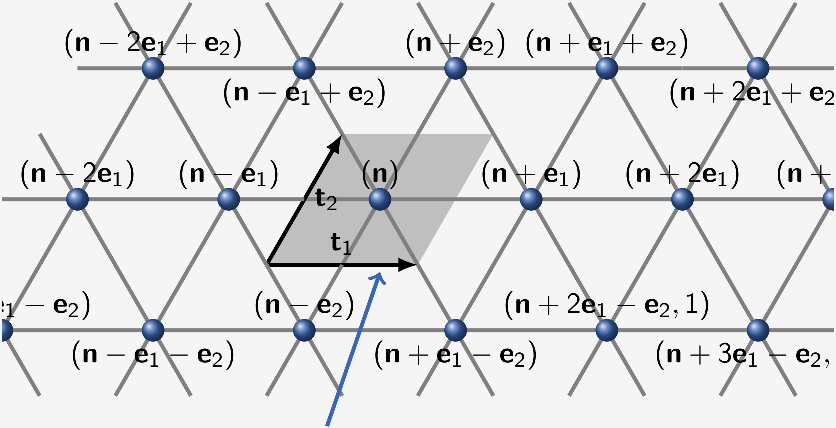
$$\mathbf{x}^{(\mathbf{n})} = \mathbf{n} \otimes \mathcal{T} \quad , \quad \mathcal{T} = [\mathbf{t}_1, \mathbf{t}_2] \quad , \quad \mathbf{t}_1 = [\ell, 0]^T \; , \; \mathbf{t}_2 = \ell[1, \sqrt{3}]^T/2$$



cell $\mathbf{n} = [n_1, n_2]^\mathsf{T} \in \mathbb{Z}^2$ $\qquad \mathbf{e}_i = [\delta_{1i}, \delta_{2i}]^\mathsf{T}$

Uniform Triangular Lattice

$$\mathbf{x}^{(\mathbf{n})} = \mathbf{n} \otimes \mathcal{T} \quad , \quad \mathcal{T} = [\mathbf{t}_1, \mathbf{t}_2] \quad , \quad \mathbf{t}_1 = [\ell, 0]^T \; , \; \mathbf{t}_2 = \ell[1, \sqrt{3}]^T/2$$



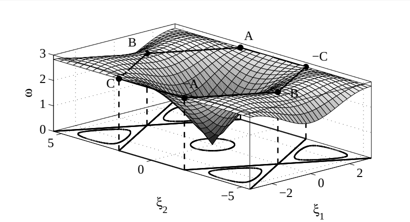
cell $\mathbf{n} = [n_1, n_2]^\mathrm{T} \in \mathbb{Z}^2 \qquad \mathbf{e}_i = [\delta_{1i}, \delta_{2i}]^\mathrm{T}$

Uniform Triangular Lattice

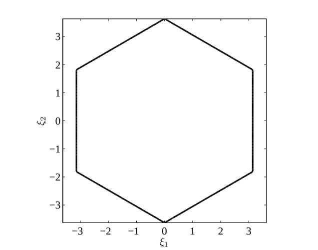
Dispersive properties

Fourier Transform:

$$\underbrace{\left[\omega^2 - 6 + 2\cos\xi_1 + 4\cos(\xi_1/2)\cos(\xi_2\sqrt{3}/2)\right]}_{\sigma(\omega,\boldsymbol{\xi})}\mathcal{U}(\boldsymbol{\xi}) = 1,$$



$$\sigma(\omega,\boldsymbol{\xi}) = 0$$

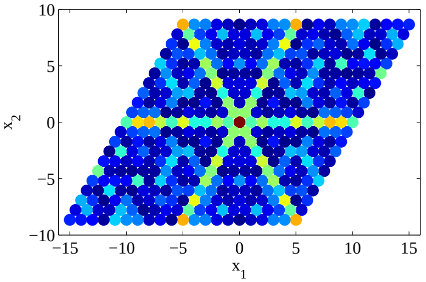


$$\sigma(2\sqrt{2},\boldsymbol{\xi}) = 0$$

Uniform Triangular Lattice

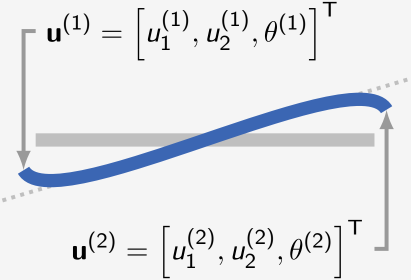
Green's function for time-harmonic out-of-plane shear

$$u(\mathbf{n}) = \frac{\sqrt{3}}{4\pi^2} \int_0^{2\pi} d\xi_1 \int_0^{2\pi/\sqrt{3}} d\xi_2 \, \frac{\cos[(n_1 + n_2/2)\xi_1] \cos[n_2 \xi_2 \sqrt{3}/2]}{\sigma(\omega, \boldsymbol{\xi})}$$



In-plane vector elasticity: Equilibrium equations

Displacement amplitude field for time-harmonic waves



- $F_1^{(i)} = ES\partial_x u_1(x)$,
- $F_2^{(i)} = -EI\partial_{xxx}^3 u_2(x)$,
- $F_3^{(i)} = -EI\partial_{xx}^2 u_2(x)$.

- The axial deformation satisfies

$$(\partial_{xx}^2 - \omega^2\rho\ell/(ES)u_1(x) = 0,$$

with $u_1(0) = u_1^{(1)}, u_1(\ell) = u_1^{(2)}$.

- The flexural deformation solves

$$[\partial_{xx}^4 - \omega^2\rho S/(EI)]u_2(x) = 0,$$

with $u_2(0) = u_2^{(1)},\ u_2(\ell) = u_2^{(2)}$, $\partial_{xx}^2 u_2|_{x=0} = \theta^{(1)},\ \partial_{xx}^2 u_2|_{x=\ell} = \theta^{(2)}$.

- Inclusion of flexural deformation allows for micropolar rotations.

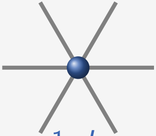
$$\mathbf{F}^{(1)} = \overbrace{\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{23} & a_{33} \end{bmatrix}}^{A} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ \theta^{(1)} \end{bmatrix} + \overbrace{\begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & b_{23} \\ 0 & -b_{23} & b_{33} \end{bmatrix}}^{B} \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ \theta^{(2)} \end{bmatrix}$$

$$\begin{aligned}
a_{11} &= \eta \cot \eta \\
a_{22} &= \beta\lambda^3 \left(\cosh\lambda \sin\lambda - \cos\lambda \sinh\lambda\right) / \left[2\left(1 - \cos\lambda \cosh\lambda\right)\right] \\
a_{23} &= \beta\lambda^2 \sin\lambda \sinh\lambda) / \left[2\left(1 - \cos\lambda \cosh\lambda\right)\right] \\
a_{33} &= \beta\lambda \left(\sin\lambda \cosh\lambda - \cos\lambda \sinh\lambda\right)) / \left[2\left(1 - \cos\lambda \cosh\lambda\right)\right] \\
b_{11} &= -\eta \csc \eta \\
b_{23} &= \beta\lambda^2 \left(\cosh\lambda - \cos\lambda\right) / \left[2\left(1 - \cos\lambda \cosh\lambda\right)\right] \\
b_{33} &= \beta\lambda \left(\sin\lambda - \sinh\lambda\right) / \left[2\left(\cos\lambda \cosh\lambda - 1\right)\right]
\end{aligned}$$

with $\beta = 2EI/(S\ell^2)$, $\eta = \omega\sqrt{\rho}$, $\lambda^4 = 2\omega^2\rho/\beta$.

Balance of Momentum

Uniform elastic triangular lattice



- Nodal displacement in the $\mathbf{n}^{\text{th}}$ cell: $\mathbf{u}^{(\mathbf{n})}$
- Introduce the matrices:

$$A(j) = R(j)^{\mathsf{T}} A R(j) \text{ and } B(j) = R(j)^{\mathsf{T}} B R(j)$$

- $R(j)$ - matrix of rotation by angle $j\pi/3$

Equations of motion

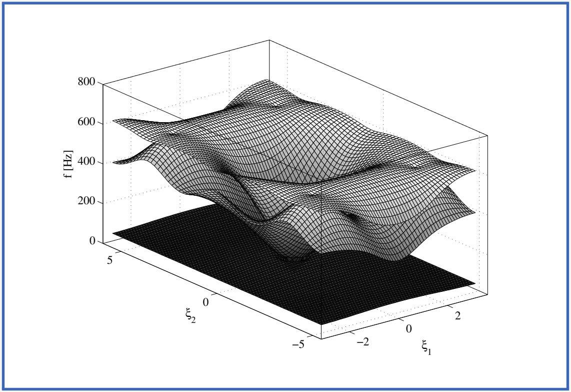
$$\begin{aligned}\omega^2 \mathbf{u}^{(\mathbf{n})} = {} & B(0)\mathbf{u}(\mathbf{n}+\mathbf{e}_1) + B(1)\mathbf{u}(\mathbf{n}+\mathbf{e}_2) + B(2)\mathbf{u}(\mathbf{n}-\mathbf{e}_1+\mathbf{e}_2) \\ & + B(3)\mathbf{u}(\mathbf{n}-\mathbf{e}_1) + B(4)\mathbf{u}(\mathbf{n}-\mathbf{e}_2) \\ & + B(5)\mathbf{u}(\mathbf{n}+\mathbf{e}_1-\mathbf{e}_2) + \sum_{0\le j\le 5} A(j)\mathbf{u}(\mathbf{n})\end{aligned}$$

Dispersion Equation

Fourier transformed equation

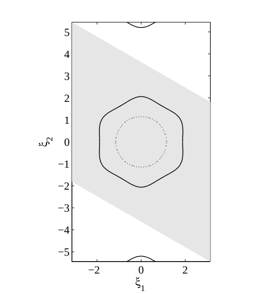
$$\omega^2 \boldsymbol{\mathcal{U}} = \sigma(\omega; \boldsymbol{\xi}) \boldsymbol{\mathcal{U}}$$

The dispersion equation is then $\det\left[\sigma(\omega, \boldsymbol{\xi}) - \omega^2 \mathbb{I}\right] = 0$.

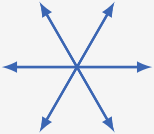


Slowness Contours

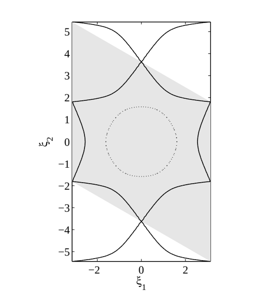
- Isotropic for small ω, as expected.
- Characteristic hexagonal contour at 323 Hz.
- Saddle point & intersection of slowness contours.
- Further saddle point at 615.8 Hz.
- "Switching" of preferential directions unique feature of elastic lattice.



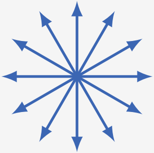
$f = 323$ Hz



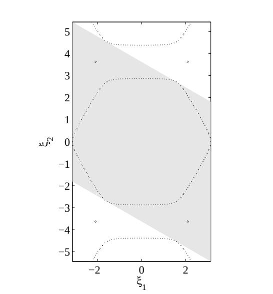
Saddle Point



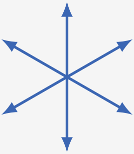
$f = 428.6$ Hz



Saddle Point



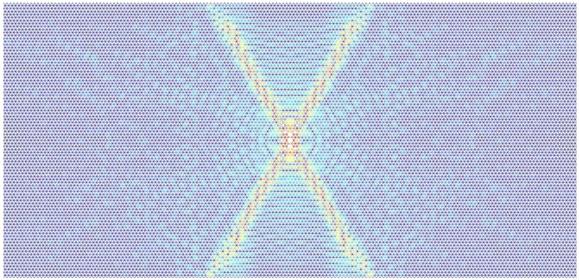
$f = 615.8$ Hz

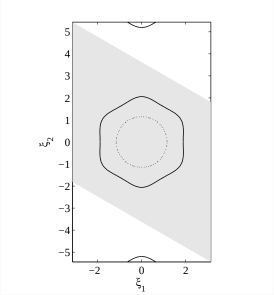


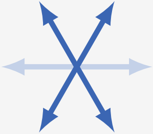
Preferential directions

Forcing Frequency of 323 Hz (Pass Band)

Horizontal Forcing

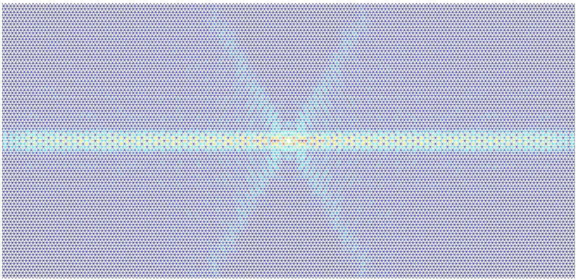


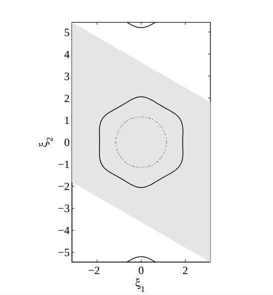


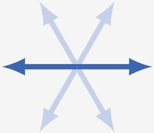


Forcing Frequency of 323 Hz (Pass Band)

Vertical Forcing

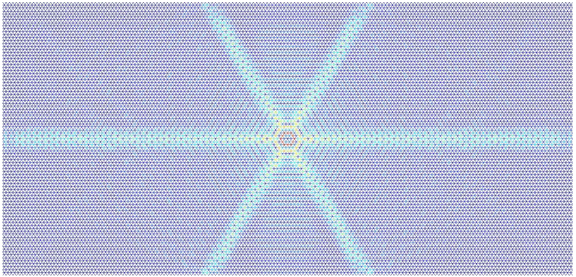


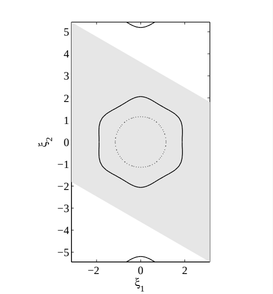


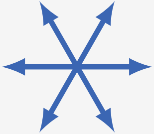


Forcing Frequency of 323 Hz (Pass Band)

Concentrated Moment

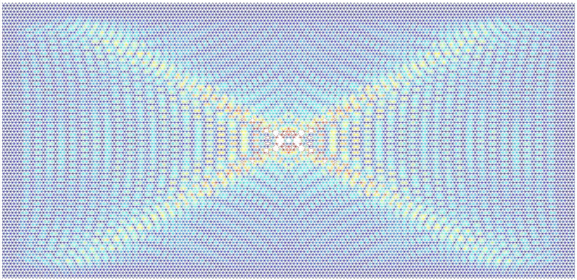


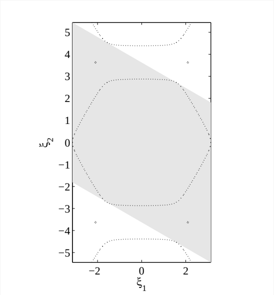


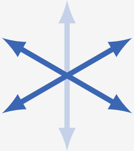


Forcing Frequency of 615.8 Hz (Saddle Point)

Horizontal Forcing

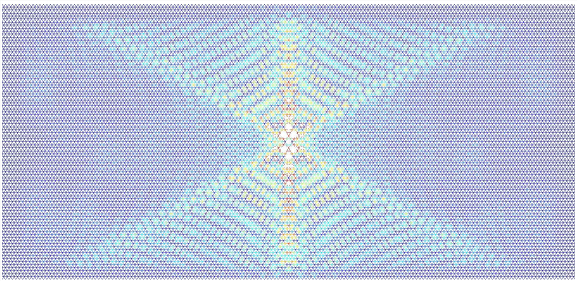


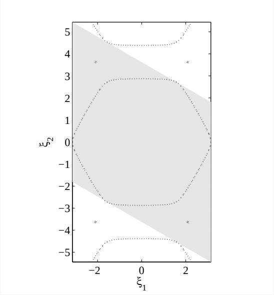


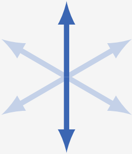


Forcing Frequency of 615.8 Hz (Saddle Point)

Vertical Forcing

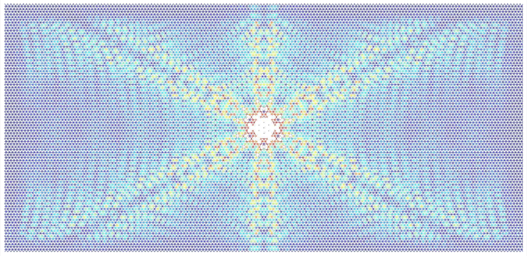


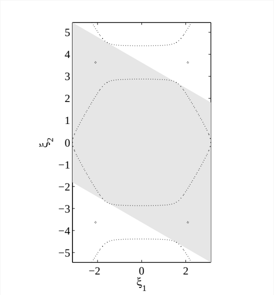


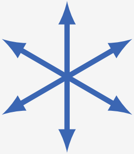


Forcing Frequency of 615.8 Hz (Saddle Point)

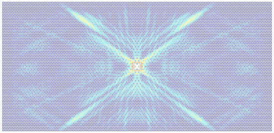
Concentrated Moment



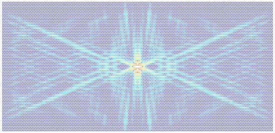




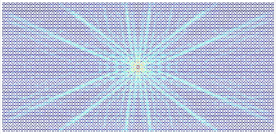
Forcing Frequency of 428.6 Hz (Saddle Point)



Horizontal Forcing



Vertical Forcing



Concentrated Moment

- Shape of primitive waveform determined by slowness contour
- Existence of primitive waveforms not associated with saddle points as in scalar case (Ayzenberg-Stepanenko et al.)
- Frequency dependent "switching" of waveform orientations for monotonic lattice is a novel feature of elastic lattice
- Elastic lattice "allows selection" of dominant orientation via type/orientation of applied forcing

Summary

- Statically isotropic lattices can exhibit very strong dynamic anisotropy
- Qualitative behaviour of fundamental solution can be predicted from the Bloch-Floquet problem
- Established the presence of primitive waveforms in elastic lattices previously only demonstrated in scalar lattice
- Offered an explanation for these effects via analysis of slowness contours
- Established the importance of the slowness contours in these primitive waveforms
- Inclusion of micropolar interactions allows for additional primitive waveforms
- Additional details can be found in
  Colquitt et al. Dynamic anisotropy & localization in elastic lattice systems. *Waves in Random & Complex Media*. To appear (doi: 10.1080/17455030.2011.633940).