Non-singular cloaking of a square inclusion with a microstructured coating

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DJC gratefully acknowledges the support of an EPSRC research studentship (EP/H018514/1). ISJ acknowledges financial support from the EPSRC (EP/H018239/1). ABM, NVM, RCM and MB acknowledge the financial support of the European Community's Seven Framework Programme under contract numbers PIAP-GA-2011-286110-INTERCER2,PIAPP-GA-284544-PARM-2 and PIEF-GA-2011-302357.

Cloaking via transformational optics

The two initiating papers

- Pendry JB, Schurig D, Smith DR. 2006 Controlling electromagnetic fields. *Science* **312**, 1780–1782
- Leonhardt U. 2006 Optical conformal mapping. *Science* **312**, 1777–1780.

Experimental validation for EM and flexural waves

- Schurig D, Mock JJ, Justice BJ, Cummer SA, Pendry JB, Starr AF, Smith DR. 2006 Metamaterial electromagnetic cloak at microwave frequencies. *Science* **314**, 977–980.
- Stenger N, Wilhelm M, Wegener M. 2012 Experiments on elastic cloaking in thin plates. *Physical Review Letters* **108**, 14301.

Cloaking via transformational optics

Cloaking in acoustics and elasticity (flexural and in-plane)

- Milton GW, Briane M, Willis JR. 2006 On cloaking for elasticity and physical equations with a transformation invariant form. *New Journal of Physics* **8**, 248.
- Norris AN. 2008 Acoustic cloaking theory. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science* **464**, 2411–2434.
- Farhat M, Guenneau S, Enoch S. 2009 Ultrabroadband elastic cloaking in thin plates. *Physical Review Letters* **103**, 24301.
- Brun M, Guenneau S, Movchan AB. 2009 Achieving control of in-plane elastic waves. *Applied Physics Letters* **94**, 061903–061903.s
- Norris AN, Shuvalov AL. 2011 Elastic cloaking theory. *Wave Motion* **48**, 525–538.

The governing equations

Consider the out-of-plane deformation u (equivalently a TE/TM polarised EM wave, or the acoustical pressure) of a thin elastic membrane.

$$\left(\nabla_{\boldsymbol{X}} \cdot \mu \nabla_{\boldsymbol{X}} + \rho\omega^2\right) u(\boldsymbol{X}) = 0, \;\; \boldsymbol{X} \in \mathbb{R}^2.$$

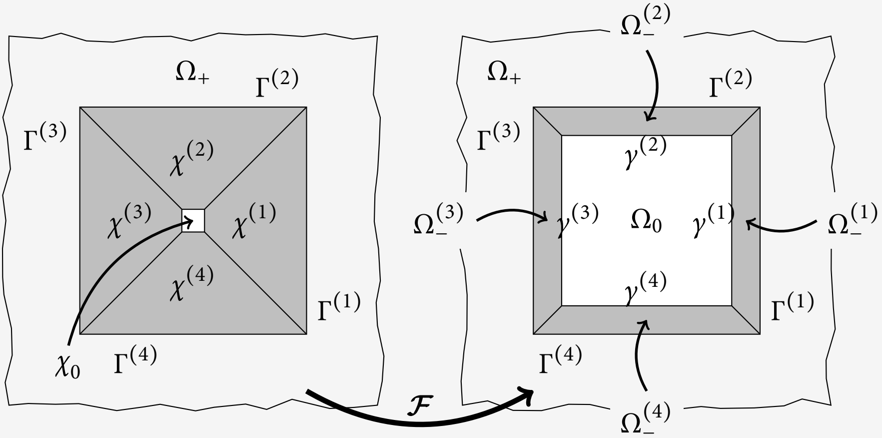
Under a mapping $\boldsymbol{x} = \mathcal{F}(\boldsymbol{X})$ the equation of motion transforms to (Norris 2008, *Proc R Soc A*, **464**)

$$\left(\nabla_{\boldsymbol{x}} \cdot \mu \boldsymbol{C}(\boldsymbol{x}) \nabla_{\boldsymbol{x}} + \frac{\rho\omega^2}{J(\boldsymbol{x})}\right) u(\boldsymbol{x}) = 0,$$

where

$$\boldsymbol{C} = \frac{\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}}{J}, \quad F_{ij} = \frac{\partial x_i}{\partial X_j}, \quad J = \det \boldsymbol{F}.$$

The regularised transformation



Similar singular transformation presented in
Rahm M et al. 2008 *Photonic Nanostruct* **6**, 87–95.

The regularised transformation

The mapping is continuous on χ, and defined in a piecewise fashion such that $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}^{(i)}(\boldsymbol{X})$ for $\boldsymbol{X} \in \chi^{(i)}$ and $\boldsymbol{\mathcal{F}}^{(i)} \in \mathcal{C}^\infty(\chi^{(i)})$, where for example,

$$\boldsymbol{\mathcal{F}}^{(1)}(\boldsymbol{X}) = \begin{pmatrix} \alpha_1 X_1 + \alpha_2 \\ \alpha_1 X_2 + \alpha_2 X_2/X_1 \end{pmatrix},$$

$\alpha_1 = w/(a + w - \varepsilon)$ and
$\alpha_2 = (a + w)(a - \varepsilon)/(a + w - \varepsilon)$,
$0 < \varepsilon \ll 1$;

$$\boldsymbol{F}^{(1)} = \begin{pmatrix} \alpha_1 & 0 \\ \dfrac{x_2\alpha_1\alpha_2}{x_1(\alpha_2 - x_1)} & \dfrac{x_1\alpha_1}{x_1 - \alpha_2} \end{pmatrix}$$

$$J^{(1)} = \frac{x_1\alpha_1^2}{x_1 - \alpha_2}$$

The material properties

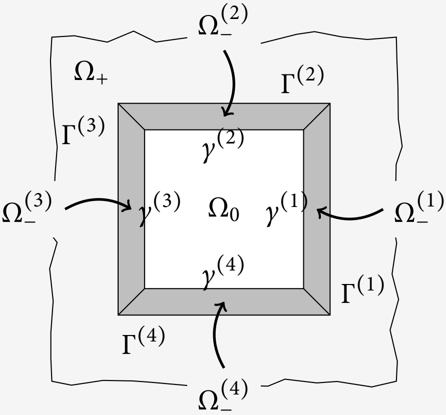
Stiffness for $\boldsymbol{x} \in \Omega_1$:
$C^{(1)} = \mu \boldsymbol{F}^{(1)} \boldsymbol{F}^{(1)^{\mathrm{T}}} [J^{(1)}]^{-1}$

$$\boldsymbol{C}^{(1)} = \mu \begin{pmatrix} \dfrac{x_1 - \alpha_2}{x_1} & -\dfrac{\alpha_2 x_2}{x_1^2} \\ -\dfrac{\alpha_2 x_2}{x_1^2} & \dfrac{x^4 + \alpha_2^2 x_2^2}{x_1^3 (x_1 - \alpha_2)} \end{pmatrix}$$

Density for $\boldsymbol{x} \in \Omega_1$

$$\frac{\rho}{J} = \rho \frac{x_1 - \alpha_2}{\alpha_1^2 x_1^2}$$

where μ and ρ are the stiffness and density in Ω^+.



Interface conditions

Without loss of generality, consider just a single side of the cloak $\Omega^{(1)}$ embedded in $\mathbb{R}^2$ and introduce

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{cases} \boldsymbol{C}^{(1)}(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \Omega_-^{(1)} \\ \mu\mathbb{I} & \text{for } \boldsymbol{x} \in \Omega_+ \end{cases}, \qquad \rho(\boldsymbol{x}) = \begin{cases} \rho(x_1 - \alpha_2)/(\alpha_1^2 x_1^2) & \text{for } \boldsymbol{x} \in \Omega_-^{(1)} \\ \rho & \text{for } \boldsymbol{x} \in \Omega_+ \end{cases}$$

together with the Helmholtz operator $\mathcal{L} = \nabla \cdot (\boldsymbol{A}(\boldsymbol{x})\nabla) + \rho(\boldsymbol{x})\omega^2$.

Let $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ be piecewise smooth solutions of the Helmholtz equation in $\mathbb{R}^2$ satisfying the Sommerfeld radiation condition at infinity.

Interface conditions

Integrating the difference $u(\boldsymbol{x})\mathcal{L}v(\boldsymbol{x}) - v(\boldsymbol{x})\mathcal{L}u(\boldsymbol{x})$ over a disc $\mathcal{D}_r$ of radius r containing $\Omega_-^{(1)}$ yields

$$
\begin{aligned}
0 &= \int_{\mathcal{D}_r} \left(u\nabla\cdot\boldsymbol{A}\nabla v - v\nabla\cdot\boldsymbol{A}\nabla u\right)\mathrm{d}\boldsymbol{x}, \\
&= \int_{\partial\Omega_-^{(i)}} \left(u^-\boldsymbol{n}\cdot\boldsymbol{A}\nabla v^- - v^-\boldsymbol{n}\cdot\boldsymbol{A}\nabla u^-\right)\mathrm{d}\boldsymbol{x} - \int_{\partial\Omega_-^{(i)}} \left(u^+\boldsymbol{n}\cdot\boldsymbol{A}\nabla v^+ - v^+\boldsymbol{n}\cdot\boldsymbol{A}\nabla u^+\right)\mathrm{d}\boldsymbol{x} \\
&\qquad + \mu\int_{\partial\mathcal{D}_r} \left(u\boldsymbol{n}\cdot\nabla v + v\boldsymbol{n}\cdot\nabla u\right)\,\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Hence, the essential interface condition is

$$[u] = 0 \qquad \text{on} \qquad \partial\Omega_-^{(1)},$$

and the natural interface condition is

$$\boldsymbol{n}\cdot\boldsymbol{C}^{(1)}\nabla u^- = \mu\boldsymbol{n}\cdot\nabla u^+ \qquad \text{on} \qquad \partial\Omega_-^{(1)}.$$

The cloaking problem

Consider the propagation of time harmonic out-of-plane deformations, generated by a point source, in a homogeneous infinite elastic solid in which is embedded an inclusion surrounded by a cloak.

$$[\nabla \cdot \boldsymbol{A}(\boldsymbol{x})\nabla + \rho(\boldsymbol{x})\omega^2]u(\boldsymbol{x}) = -\delta(\boldsymbol{x} - \boldsymbol{x}_0), \qquad \boldsymbol{x} \in \mathbb{R}^2 \setminus \bar{\Omega}_0, \qquad \boldsymbol{x}_0 \in \Omega_+$$

$$[\nabla \cdot \mu_0 \nabla + \varrho_0 \omega^2]u(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \Omega_0,$$

with continuity of $u(\boldsymbol{x})$ and tractions on all internal boundaries Additionally, the Sommerfeld radiation condition is imposed at infinity. The stiffness tensor $\boldsymbol{A}(\boldsymbol{x})$ and density $\rho(\boldsymbol{x})$ are

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{cases} \boldsymbol{C}^{(i)}(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \Omega_-^{(i)} \\ \mu\mathbb{I} & \text{for } \boldsymbol{x} \in \Omega_+ \end{cases}, \qquad \rho(\boldsymbol{x}) = \begin{cases} \rho\{J^{(i)}(\boldsymbol{x})\}^{-1} & \text{for } \boldsymbol{x} \in \Omega_-^{(i)} \\ \rho & \text{for } \boldsymbol{x} \in \Omega_+ \end{cases},$$

and μ_0 and ρ_0 are the stiffness and density of the inclusion respectively.

The ray equations

Consider a WKB expansion of the displacement amplitude field

$$u(\boldsymbol{x}) \sim e^{i\omega\phi(\boldsymbol{x})} \sum_{n=0}^{\infty} \frac{i^n U_n(\boldsymbol{x})}{\omega^n}, \qquad \text{as } \omega \to \infty,$$

whence the leading order equation for the phase on the interior of the cloak is

$$H(\boldsymbol{x}, \boldsymbol{s}) = \mu\rho^{-1}\boldsymbol{s}\cdot\boldsymbol{g}^{-1}\boldsymbol{s} - 1 = 0,$$

where $\boldsymbol{s} = \nabla\phi$ and $\boldsymbol{g} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$.
Characteristics:

$$\frac{\mathrm{d}H}{dt} = 0, \qquad \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\partial H}{\partial \boldsymbol{s}}, \qquad \frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = -\frac{\partial H}{\partial \boldsymbol{x}},$$

where t is the ray/time parameter. In index summation notation

$$\boxed{\frac{\mathrm{d}s_i}{\mathrm{d}t} = -2\rho^{-1}\mu s_m s_n F_{nl}\frac{\partial F_{ml}}{\partial x_i}, \qquad \frac{\mathrm{d}x_i}{\mathrm{d}t} = 2\rho^{-1}\mu F_{il}F_{jl}s_j.}$$

The ray equations

Alternative representation of Hamiltonian

$$\tilde{H}(\boldsymbol{x}, \boldsymbol{n}) = \mu\rho^{-1}\boldsymbol{n}\cdot\boldsymbol{g}^{-1}\boldsymbol{n} - v^2 = 0,$$

where $\boldsymbol{s} = \boldsymbol{n}/v = \boldsymbol{n}/|\boldsymbol{F}^{\mathrm{T}}\boldsymbol{n}|\sqrt{\rho/\mu}$.
Consider a ray in the ambient medium, in direction $\boldsymbol{N}$ passing through $\boldsymbol{X}_0$. In the cloak, the corresponding curve is $\boldsymbol{x}(t) = \mathcal{F}(\boldsymbol{X}_0 + t\boldsymbol{N})$. Hence,

$$\boxed{\frac{\mathrm{d}x_i}{\mathrm{d}t} = F_{ij}N_j = F_{il}F_{jl}s_j\sqrt{\frac{\mu}{\rho}}.}$$

Taking the derivative of $\boldsymbol{s}$ for constant N and using the compatibility condition that the deformation gradient should be irrotational under finite deformation $\varepsilon_{jk\ell}\partial J^{-1}{}_{ik}/\partial x_j = 0_{\ell i}$

$$\boxed{\frac{\mathrm{d}s_i}{\mathrm{d}t} = -s_m s_n F_{n\ell}\frac{\partial F_{ml}}{\partial x_i}\sqrt{\frac{\mu}{\rho}}.}$$

The ray equations

Rays on the interior of the cloak:

- From the Hamiltonian

$$\boxed{\frac{\mathrm{d}s_i}{\mathrm{d}t} = -2\rho^{-1}\mu s_m s_n F_{nl}\frac{\partial F_{ml}}{\partial x_i}, \qquad \frac{\mathrm{d}x_i}{\mathrm{d}t} = 2\rho^{-1}\mu F_{il}F_{jl}s_j.}$$

- From transforming a straight line in the ambient medium

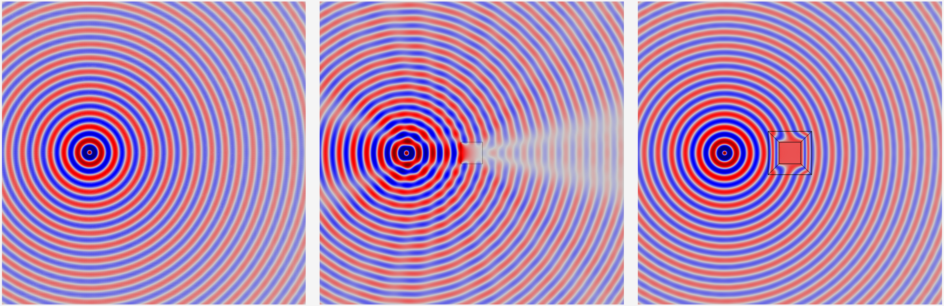
$$\boxed{\frac{\mathrm{d}s_i}{\mathrm{d}t} = -s_m s_n F_{n\ell}\frac{\partial F_{ml}}{\partial x_i}\sqrt{\frac{\mu}{\rho}}, \qquad \frac{\mathrm{d}x_i}{\mathrm{d}t} = F_{ij}N_j = F_{il}F_{jl}s_j\sqrt{\frac{\mu}{\rho}}.}$$

Hence, rays in the cloak are simply straight lines deformed according to the mapping $\mathcal{F}$.

Ray paths

Numerical simulations

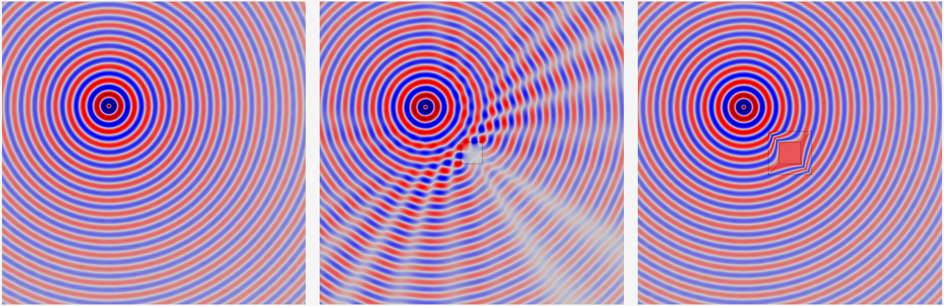
$\omega = 10$



Intact Uncloaked Cloaked

Numerical simulations

$\omega = 10$



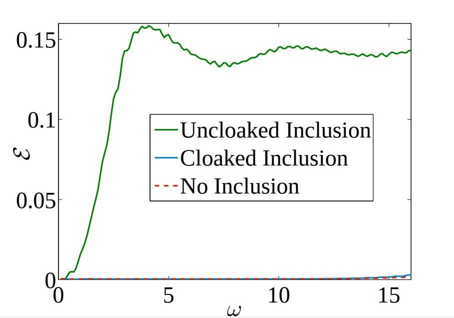
Intact Uncloaked Cloaked

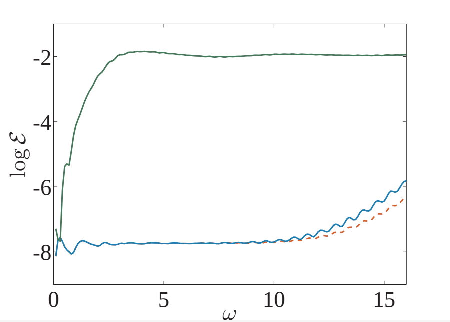
Numerical simulations

Scattering measure

$$\mathcal{E}(u_1, u_2, \mathcal{R}) = \left(\int_{\mathcal{R}} |u_1(\boldsymbol{x}) - u_2(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} \right) \left(\int_{\mathcal{R}} |u_2(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} \right)^{-1}$$

$\mathcal{E} = 0$ corresponds to perfect cloaking with no numerical error.





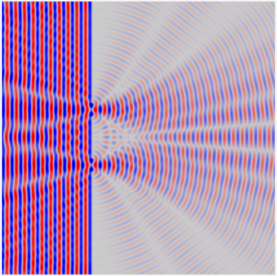
Scattering measure

Typical values of $\mathcal{E}$ are

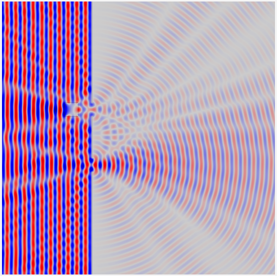
Source Position	Source Frequency	Scattering Measure $\mathcal{E}$ Uncloaked	Scattering Measure $\mathcal{E}$ Cloaked	Q
$[-3, 0]^{\mathrm{T}}$	5	0.1529	4.351×10^{-4}	0.9972
$[-3, 0]^{\mathrm{T}}$	10	0.1455	4.514×10^{-4}	0.9969
$[-3, 3]^{\mathrm{T}}/\sqrt{2}$	5	0.2002	3.941×10^{-4}	0.9980
$[-3, 3]^{\mathrm{T}}/\sqrt{2}$	10	0.3286	4.068×10^{-4}	0.9988

where

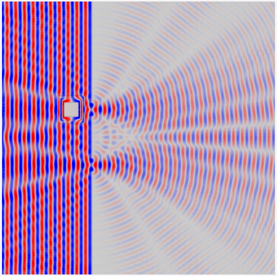
- Uncloaked: $\mathcal{E}(u_{\text{uncloaked}}, u_{\text{GF}})$
- Cloaked: $\mathcal{E}(u_{\text{cloaked}}, u_{\text{GF}})$
- $Q = 1 - \mathcal{E}(u_{\text{cloaked}}, u_{\text{GF}})/\mathcal{E}(u_{\text{uncloaked}}, u_{\text{GF}})$



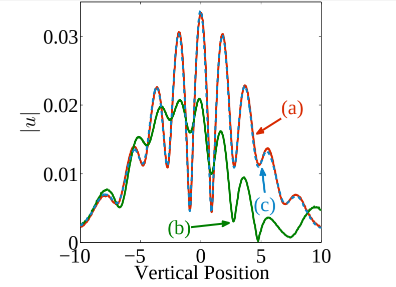
(a) No inclusion



(b) Uncloaked Inclusion



(c) Cloaked Inclusion



Cloaked inclusion

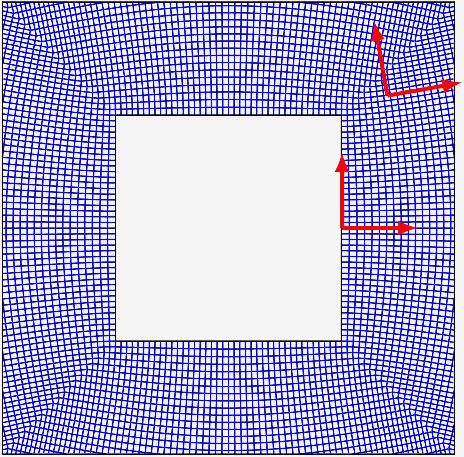
Uncloaked inclusion

Cloaking with a lattice - geometry

The symmetric stiffness matrices $\boldsymbol{C}^{(i)} = [\mu/J^{(i)}]\boldsymbol{F}^{(i)^{\mathrm{T}}}\boldsymbol{F}^{(i)}$ are positive definite, hence

$$\boldsymbol{C}^{(i)} = \boldsymbol{P}^{(i)^{\mathrm{T}}}\boldsymbol{\Lambda}^{(i)}\boldsymbol{P}^{(i)},$$

- $\boldsymbol{P}^{(i)} = [\boldsymbol{e}_1^{(i)}, \boldsymbol{e}_2^{(i)}]$
- $\boldsymbol{e}_1^{(i)}$ and $\boldsymbol{e}_2^{(i)}$ - eigenvectors of $\boldsymbol{C}^{(i)}$
- $\boldsymbol{\Lambda}^{(i)} = \mathrm{diag}(\lambda_1^{(i)}, \lambda_2^{(i)})$
- $0 < \lambda_2^{(i)} < \lambda_1^{(i)}$ - eigenvalues of $\boldsymbol{C}^{(i)}$



Cloaking with a lattice - material properties

Lattice nodes lie at the intersection points of the characteristics

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{x}_j^{(i)} = \boldsymbol{e}_j^{(i)}(\boldsymbol{x}_j^{(i)}),$$

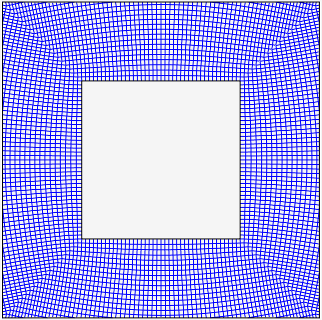
for $i = 1, \ldots 4,$ and $j = 1, 2.$ The nodal mass is obtained by evaluating the integral

$$m(\boldsymbol{x_p}) = \int_{\mathcal{A}(\boldsymbol{x_p})} \rho(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

over the unit cell $\mathcal{A}(\boldsymbol{x_p})$.

Requiring local conservation of flux yields the stiffness of the link along $\boldsymbol{e}_j^{(i)}$

$$k_{ij} = \ell_{ij}\lambda_j^{(i)}.$$



Approximate lattice

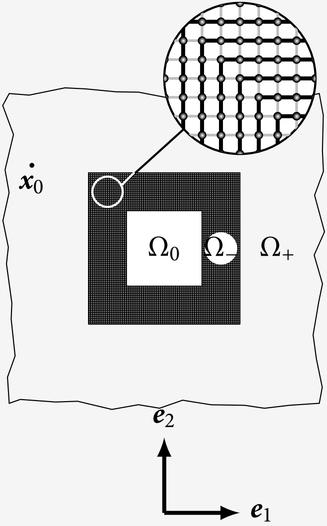
For $w/a \ll 1$ $\boldsymbol{e}_j^{(i)} \approx [\delta_{i1}, \delta_{i2}]^{\mathrm{T}}$ and the lattice may be approximated by a square lattice.

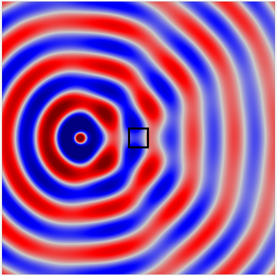
$$[\nabla \cdot \mu \nabla + \varrho \omega^2] u(\boldsymbol{x}) = -\delta(\boldsymbol{x} - \boldsymbol{x}_0), \ \boldsymbol{x}, \boldsymbol{x}_0 \in \Omega_+,$$

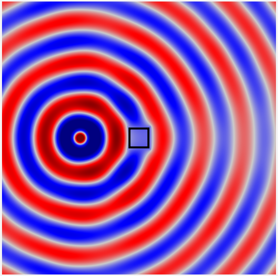
$$[\nabla \cdot \mu_0 \nabla + \varrho_0 \omega^2] u(\boldsymbol{x}) = 0, \ \boldsymbol{x} \in \Omega_0,$$

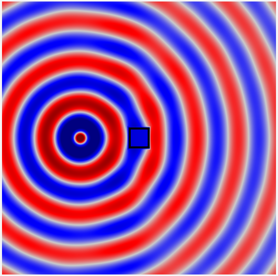
$$0 = m(\boldsymbol{p}) \omega^2 u(\boldsymbol{p}) + \sum_{\boldsymbol{q} \in \mathcal{N}(\boldsymbol{p})} \ell \eta(\boldsymbol{q}, \boldsymbol{p}) \left[u(\boldsymbol{p} + \boldsymbol{q}) - u(\boldsymbol{p}) \right], \text{ in } \Omega_-,$$

where $\boldsymbol{e}_i = [\delta_{i1}, \delta_{i2}]^{\mathrm{T}}$, $\boldsymbol{p} \in \mathbb{Z}^2$, and $\mathcal{N} = \{\pm \boldsymbol{e}_1, \pm \boldsymbol{e}_2\}$, and $\ell \eta(\boldsymbol{q}, \boldsymbol{p})$ is the stiffness of the link connecting nodes $\boldsymbol{p}$ and $\boldsymbol{q}$.





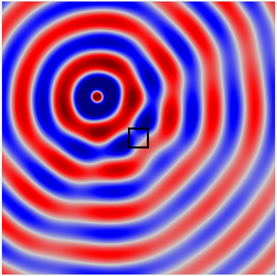


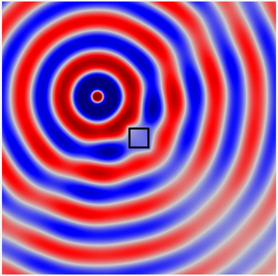


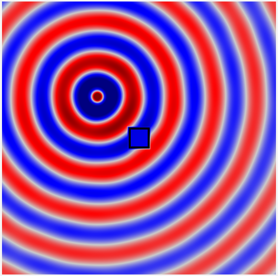
Uncloaked
$\omega = 3$

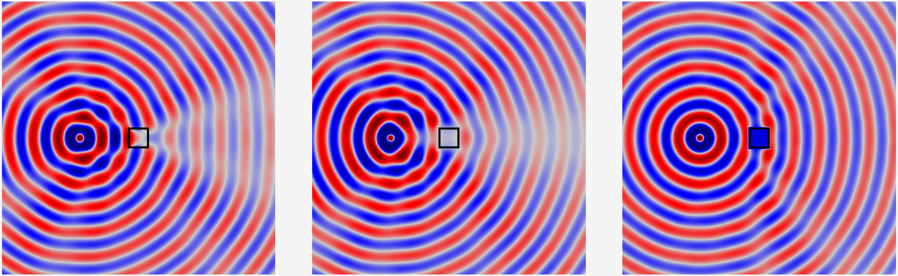
Basic lattice cloak
Constant Stiffness

Refined lattice cloak
Variable stiffness





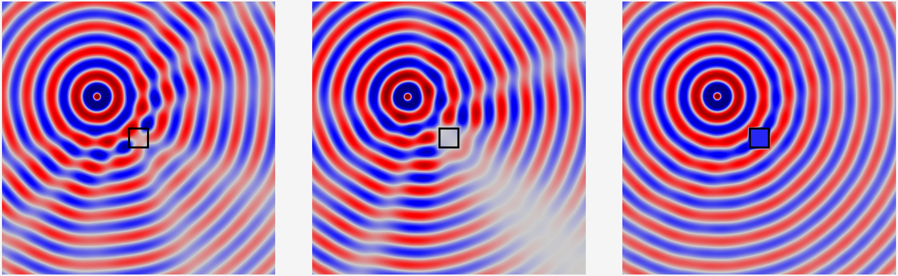




Uncloaked
$\omega = 5$

Basic lattice cloak
Constant Stiffness

Refined lattice cloak
Variable stiffness



Scattering measure

Typical values of $\mathcal{E}$ are

Source		Scattering Measure $\mathcal{E}$		
Position	Frequency	Uncloaked	Cloaked	Q
$[-3,0]^{\mathrm{T}}$	3	0.1430	0.01191	0.8929
$[-3,3]^{\mathrm{T}}/\sqrt{2}$	3	0.1113	3.385×10^{-3}	0.9763
$[-3,0]^{\mathrm{T}}$	5	0.1529	0.04324	0.7173
$[-3,3]^{\mathrm{T}}/\sqrt{2}$	5	0.2002	0.03125	0.8438

where

- Uncloaked: $\mathcal{E}(u_{\text{uncloaked}}, u_{\text{GF}})$
- Cloaked: $\mathcal{E}(u_{\text{cloaked}}, u_{\text{GF}})$
- $Q = 1 - \mathcal{E}(u_{\text{cloaked}}, u_{\text{GF}})/\mathcal{E}(u_{\text{uncloaked}}, u_{\text{GF}})$

Concluding remarks

- Analysed a regularised invisibility cloak for a square inclusion and elastic waves in a membrane (or EM or acoustic waves)
- The metamaterial cloak has non-singular and piecewise smooth material properties
- Wave propagation through the cloak was analysed via ray equations derived via a WKB approximation as well as full wave numerical simulations
- Examined the efficacy of the cloak using novel techniques and demonstrated that the cloak is effective over a wide frequency range
- The geometry of the cloak allows for a straightforward connection to be made with a microstuctured lattice coating
- Designed metamaterial cloaks using a simple mass-spring lattice system, which may allow a practical implementation
- Demonstrated that such a lattice cloak is efficient in the low frequency regime

Thank you for your attention

Further details may be found in the pre-print

Making Waves Round a Structured Cloak: Lattices, Negative Refraction and Fringes *Proc R Soc A*, in press.
ArXiV preprint **1304.1365** http://arxiv.org/abs/1304.1365