ASYMPTOTIC METHODS IN MECHANICS

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Contents

Introduction

The module *Asymptotic Methods in Mechanics* consists of analytical approximation techniques for solving algebraic, ordinary and partial differential equations and corresponding initial and boundary value problems with a small parameter arising in mechanics and other areas of sciences, engineering and technology.

It contains 18 lectures and 4 tutorials dealing with the course assignments. Students are encouraged to seek help from this Lecture Notes and the following literature: "Perturbation Methods" by E.J. Hinch and "Perturbation Methods" by A.H. Nayfeh. Also, the book "Perturbation Methods in Applied Mathematics" by J. Kevorkian and J.D. Cole is recommended for reading as well as the book "Asymptotic analysis of Differential Equations" by R.B. White published by Imperial College press. We will need Chapter 1 on Dominant Balance from this book. This chapter is also available from Internet as well as a useful textbook "Asymptotic Analysis and Singular Perturbation Theory" by J.K. Hunter.

The authors are thankful to Prof. A. Movchan (Department of Mathematical Sciences, University of Liverpool) for remarks, suggestions and comments for the first part of the course. The authors also would like to express their gratitude to Prof. A. Korobkin (School of Mathematics, University of East Anglia) for providing the notes of his lectures on Asymptotic Analysis read for the third year and Master students at UEA in 2008 and 2010. These notes were partially used in Chapter 1. The last two chapters of this Lecture Notes are based on the textbook [5].

In this module we give mathematical definitions of asymptotic expansions, asymptotic formulas and shall study:

1) Operations with asymptotic expansions;

2) Perturbation methods for algebraic equations with small parameter;

3) Regular and singular perturbation methods for ordinary differential equations;

4) Regular perturbation methods for boundary problems in partial differential equations;

5) Matched asymptotic expansions: outer and inner representations;

6) Van Dyke's matching rule and composite approximations;

7) Homogenization method;

8) Method of multiple scales;

9) Method of asymptotic expansions in thin domains.

Chapter 1

Asymptotic expansions

1.1 Introduction to asymptotic analysis

1.1.1 Mathematical formulation of a physical problem

We will deal mainly with constructing the so-called asymptotic (approximate) solutions of mathematical problems written in non-dimensional variables with only occasional regard for the physical situation it represents. However, to start things off it is worth considering a typical physical problem to illustrate where the mathematical problems originate. A simple example comes from the motion of an object projected radially upward from the surface of the Earth. The object moves due to its initial velocity and the gravitational force given by Newton's law of universal gravitation. This law is an empirical physical law describing the gravitational attraction between bodies

$$F = -\frac{\gamma m M}{r^2},$$

where r is the distance between the centers of mass of two bodies with masses m and M. In our problem, m is the mass of the object, M is the mass of the Earth, and γ is the gravitational constant, $\gamma = 6.67428 \times 10^{-11}\ \mathrm{m}^3\,\mathrm{kg}^{-1}\mathrm{s}^{-2}$.

Let $x(t)$ denote the height of the object, measured from the surface of the Earth. Then from Newton's second law, we obtain the following equation of motion:

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(R+x)^2}, \quad t > 0, \tag{1.1}$$

where R is the radius of the Earth and g is the gravitational constant defined as $\gamma M R^{-2}$. We assume that the object starts from the surface with a given upward (positive) velocity v_0, i. e.,

$$x(0) = 0, \quad x'(0) = v_0. \tag{1.2}$$

The nonlinear nature of the ordinary differential Eq. (1.1) makes finding a closed-form solution difficult, and it is thus natural to try searching for a way to simplify it. For example, if the object does not get far from the surface, then one might try to argue that $x(t)$ is small compared to R and the denominator in (1.1) can be simplified to R^2. This is the type of argument often made in introductory physics and engineering texts. In this case $x \approx x_0(t)$, where $x_0'' = -g$ for $x_0(0) = 0$ and $x_0'(0) = v_0$. It is not difficult to find the respective solution

$$x_0(t) = -\frac{1}{2}gt^2 + v_0 t. \tag{1.3}$$

One finds in this case that the object reaches a maximum height of $v_0^2/2g$ and comes back to Earth surface at time $t_* = 2v_0/g$. The difficulty with this reduction is that it is unclear how to determine a correction to the approximate solution in (1.3). This is worth knowing since we would then be able to get a measure of the error made in using (1.3) as an approximation to the unknown solution and it would also be possible to see just how the nonlinear nature of the original problem affects the motion of the object. To make the reduction process more *systematic*, it is first necessary to scale the variables and to rewrite the problem in non-dimensional variables. To do this, we put

$$\tau = t/t_c, \quad y(\tau) = x(t)/x_c, \tag{1.4}$$

where t_c is a characteristic time for the problem and x_c is a characteristic value for the projectile displacement. We have a lot of freedom in choosing these constants, but they should be *representative for the regime* we are going to investigate.

Based on the solution $x_0(t)$ of the approximate problem, we take

$$t_c = v_0/g, \quad x_c = v_0^2/g.$$

Correspondingly, we will have

$$\frac{d^2x}{dt^2} = \frac{x_c}{t_c^2}\frac{d^2y}{d\tau^2} = g\frac{d^2y}{d\tau^2}. \tag{1.5}$$

In the new variables the problem is formulated as follows:

$$\frac{d^2y}{d\tau^2} = -\frac{1}{(1+\varepsilon y)^2}, \quad \tau > 0, \tag{1.6}$$

$$y(0) = 0, \quad y'(0) = 1. \tag{1.7}$$

Note that the problem (1.1), (1.2) is equivalent to the problem (1.6), (1.7).

The parameter $\varepsilon = v_0^2/(gR)$ involved in (1.6) is now dimensionless. Moreover, its value is of crucial importance as it gives us a measure of how high the projectile can go in comparison to the radius of the Earth. We know that $R \approx 6.4 \times 10^6$ m and $g \approx 10$ m/s^{-2}. Then $\varepsilon = 1$ if one takes $v_0 = 8$ km/s. Therefore, if the initial velocity of a projectile is much less than 8 km/s, then ε in (1.6) is a small parameter and we can use *asymptotic methods* to find approximate solutions of the problem. For example, if the initial velocity is comparable with the speed of a fast Formula One car ($v = 360$ km/h or $v = 100$ m/s) then $\varepsilon = 1.56 \cdot 10^{-4}$, and so it is a pretty small value.

Everywhere in this module to set that value of the parameter ε is small enough we will use the following notation:

$$\varepsilon \ll 1.$$

Introducing non-dimensional variables we obtain non-dimensional parameters of the problem. This allow us later to compare terms in the equation while searching for the solution. On the other hand, recognising which term is more essential than others in terms of the introduced small dimensionless parameter ε, one can construct a sequence of solutions where every next solution is close to the exact one than the previous solution. This we will describe in detail in the next subsection.

1.1.2 Example 1. Asymptotic solution of a Cauchy problem for an ordinary differential equation

We assume that the solution to the problem (1.6), (1.7) can be found in the form of a power series with respect to the small parameter ε as

$$y(\tau, \varepsilon) = y_0(\tau) + \varepsilon y_1(\tau) + \varepsilon^2 y_2(\tau) + \dots . \tag{1.8}$$

We insert the two-term straightforward expansion of the solution (1.8) into Eq. (1.6) and make use of the expansion

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \dots,$$

which is valid for small values of z and can be obtained by differentiating the well known power series

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots$$

with respect to z. Setting $z = -\varepsilon y$, we find

$$\frac{1}{(1+\varepsilon y)^2} = 1 - 2\varepsilon y + 3\varepsilon^2 y^2 + \ldots .$$

Thus, Eq. (1.6) takes the form

$$y_0'' + \varepsilon y_1'' + \ldots . = -1 + 2\varepsilon(y_0 + \varepsilon y_1 + \ldots) - 3\varepsilon^2(y_0 + \varepsilon y_1 + \ldots)^2 + \ldots . \quad (1.9)$$

Since $y_n(\tau)$ are independent of ε, and the both sides of Eq. (1.9) have to be equal for any small value of the parameter ε, the coefficients of like powers of ε must be the same on both sides of this equation. Equating the respective coefficients, we will have

$$\varepsilon^0 : \qquad y_0'' = -1, \qquad (1.10)$$

$$\varepsilon^1 : \qquad y_1'' = 2y_0, \qquad (1.11)$$

and so on for higher powers of the small parameter ε.

Inserting the two-term straightforward expansion of the solution into the initial conditions (1.7) gives

$$\varepsilon^0 : \qquad y_0(0) = 0, \quad y_0'(0) = 1, \qquad (1.12)$$

$$\varepsilon^1 : \qquad y_1(0) = 0, \quad y_1'(0) = 0. \qquad (1.13)$$

Solving Eq. (1.10) with the initial conditions (1.12), we find that

$$y_0(\tau) = -\frac{1}{2}\tau^2 + \tau,$$

and this expression (after returning to the original variables) fully coincides with the approximate solution $x_0(t)$ found above in (1.3).

Equation (1.11) can be written now as

$$y_1'' = 2\tau - \tau^2,$$

and with the initial conditions (1.13) gives the following solution:

$$y_1(\tau) = \frac{\tau^3}{3} - \frac{\tau^4}{12}.$$

Finally, the so-called two term asymptotic approximation of the exact solution takes the form

$$y(\tau,\varepsilon) = -\frac{1}{2}\tau^2 + \tau + \varepsilon\left(\frac{\tau^3}{3} - \frac{\tau^4}{12}\right) + \dots . \tag{1.14}$$

The process could be continued to have the approximation function more and more accurate. However, as it will be shown later, there is no guarantee that the process will converge (to the exact solution).

In fact, we could stop our analysis at this point and move forward to the next subsection. However, it is probably worth to make a small effort to persuade ourselves even more what an efficient tool we have in our hands. Indeed, let us imagine that one wants to find not only the approximate solution (1.14) but also the maximum of the object height $y_{\max}(\varepsilon)$ which the solution reaches at the normalised time $\tau = \tau_{\max}(\varepsilon)$. For this purpose, it is necessary to solve the equation

$$y_\tau(\tau_{\max}(\varepsilon), \varepsilon) = 0 \tag{1.15}$$

and a standard way to do so is to find points where $y'(\tau) = 0$ and to select one which provides the maximum. Unfortunately, the solution of the corresponding cubic equation

$$0 = -\tau + 1 + \varepsilon\left(\tau^2 - \frac{\tau^3}{3}\right), \tag{1.16}$$

which can be found by using the closed-form Cardano's formula, is not easy at all.

However, using the same methodology as above, the answer is quite straightforward. First, let us observe, that the values $\tau_{\max}(0) = 1$ and $y_{\max}(0) = 1/2$ are trivially obtainable. One can check that, after re-normalisation (1.4), we receive the same values as discussed above $t_{\max}(0) = t_c$, $x_{\max}(0) = x_c/2$. Note that the approximate solution (1.14) makes perfect sense for the normalised time τ lying in a neighborhood of unit. Let we try to estimate the value of $\tau_{\max}(\varepsilon)$ more accurate searching for it in form:

$$\tau_{\max}(\varepsilon) = 1 + \varepsilon\tau_1 + \varepsilon^2\tau_2 + \dots .$$

Substituting this into Eq. (1.16), we immediately find that

$$\tau_{\max}(\varepsilon) = 1 + \frac{2}{3}\varepsilon + \dots .$$

and, thus, we will have

$$y_{\max}(\varepsilon) = \frac{1}{2} + \frac{1}{4}\varepsilon + \dots .$$

Finally note that the nonlinearity of Eq. (1.6) of the object motion increases the maximum height and the time to reach it.

1.2 Main idea of asymptotic analysis

1.2.1 Straightforward asymptotic expansion

The fundamental idea of asymptotic analysis (or perturbation technique) is to relate the unknown solution of original complex problem to the known simple solution of an approximate associated problem by means of simple transitions. This is a comparative method, a procedure used throughout all intellectual domains.

Consider a general equation

$$L(u) + \varepsilon N(u) = 0, \tag{2.1}$$

where $L(u) = 0$ is an equation that can be readily solved with the solution u_0. We assume that N is a bounded operator defined on a space of real-valued functions.

We need to find the solution $u(\varepsilon)$ to Eq. (2.1) in case when the positive parameter ε is small, $\varepsilon \ll 1$. It can be expected that as $\varepsilon \to 0$, we will have

$$u(\varepsilon) \to u_0.$$

Furthermore, let us assume that $L(u)$ is a linear operator,

$$L(c_1 u + c_2 v) = c_1 L(u) + c_2 L(v) \tag{2.2}$$

and the solution can be found in the form of power series

$$u(\varepsilon) = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots . \tag{2.3}$$

We do not discuss convergence of the series (2.3) at this stage.

Inserting (2.3) into (2.1) and using (2.1) we derive

$$L(u_0) + \varepsilon L(u_1) + \varepsilon^2 L(u_2) + \dots + \varepsilon N(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots) = 0. \tag{2.4}$$

Assuming that $N(u)$ is not only bounded but smooth enough operator near the point $u = u_0$, we can expand it as follows

$$N(u) = N(u_0) + A_1(u - u_0) + A_2(u - u_0)^2 + \dots, \tag{2.5}$$

where A_1 and A_2 are linear operators dependent on u_0. Inserting (2.5) into (2.4), we rewrite it in form of a sum of terms. Then, collecting the terms with the same power of ε and equating the results to zero, we find

$$\varepsilon^1 : \qquad L(u_1) = -N(u_0), \tag{2.6}$$

$$\varepsilon^2 : \qquad L(u_2) = -A_1(u_1), \tag{2.7}$$

and so on. Assuming that these non-homogeneous problems can be solved, we calculate u_1, u_2 and other coefficients in (1.7). It is possible that the obtained power series does not converge but in asymptotic analysis *divergent series* are allowed. What does this means in practical terms we will explain later. The solution (2.3) is called *the straightforward asymptotic expansion* of the solution $u(\varepsilon)$. If the solution exists it usually called *regular asymptotic expansion* of the exact solution.

It is quite clear that the equation

$$L(u) + \sum_{j=1}^{k} \varepsilon^j N_j(u) = 0 \tag{2.8}$$

can be considered in the same manner. If all the operators N_j are smooth enough near the point $u = u_0$ in the sense of Eq. (2.5), then the difference in *the asymptotic procedure* in comparison with the previous case is purely technical. Note that the case $k = \infty$ is also possible. Indeed, in the example discussed in Section 1.1, we met with such situation. In that case we have the following operators:

$$L(u) = u'' + 1, \tag{2.9}$$

defined on the space of twice-differentiable functions C^2 satisfying the initial conditions (1.7), while other operators are

$$N_j(u) = (-1)^j (j + 1) u^j. \tag{2.10}$$

It is absolutely clear that the case $k = \infty$ can be also represented in the following manner:

$$L(u) + \sum_{j=1}^{K} \varepsilon^j N_j(u) + \varepsilon^{K+1} M(u, \varepsilon) = 0, \tag{2.11}$$

where K is an arbitrary large natural value and operator M depends also on the parameter ε. In this case the choice of the number K is solely depends on us (when we will to stop the asymptotic procedure). However, it may also happen that this is the only choice. Such situation appears when all operators N_j up to the number $j = K$ are smooth near the point $u = u_0$ but the operator N_{K+1} even formally written does not posses such a property. In such case not more than K terms in the asymptotic expansion can be delivered.

All the aforementioned cases belong to the so-called *regular perturbations*. Below we show another example of that type.

1.2.2 Example 2. Asymptotics of a boundary value problem for an ordinary differential equation

Previous example of a regularly perturbed problem was related to the Cauchy problem for nonlinear ODE. Now we consider a simple perturbed boundary value problem

$$u''(x) - \varepsilon^2 u(x) = 1, \quad 0 \le x \le 1, \tag{2.12}$$

$$u(0) = 0, \; u(1) = 1, \tag{2.13}$$

where ε is assumed to be a small positive parameter.

Now, the operator $Lu = u'' - 1$ acts in the set of functions from the class of smooth functions $C^2[0,1]$ satisfying the boundary conditions (2.13). Note that $N_1 \equiv 0$ and only $N_2 = -u$ differs from zero.

The leading order asymptotic approximation (for small values of ε) u_0 satisfying the equation $Lu_0 = 0$ can be immediately computed

$$u_0(x) = \frac{1}{2}x + \frac{1}{2}x^2. \tag{2.14}$$

Continuing the process, we will have

$$u(x,\varepsilon) = \frac{1}{2}(x + x^2) + \varepsilon^2 \frac{1}{24}(x^4 + 2x^3 - 3x) + \dots . \tag{2.15}$$

On the other hand, the exact solution of (2.12), (2.13), is given by

$$u_{\text{exact}} = \frac{e^{\varepsilon x} + e^{-\varepsilon x} - 2}{2\varepsilon^2} + \frac{e^{\varepsilon x} - e^{-\varepsilon x}}{2\varepsilon^2(e^{\varepsilon} - e^{-\varepsilon})}(2 - e^{\varepsilon} - e^{-\varepsilon} + 2\varepsilon^2), \tag{2.16}$$

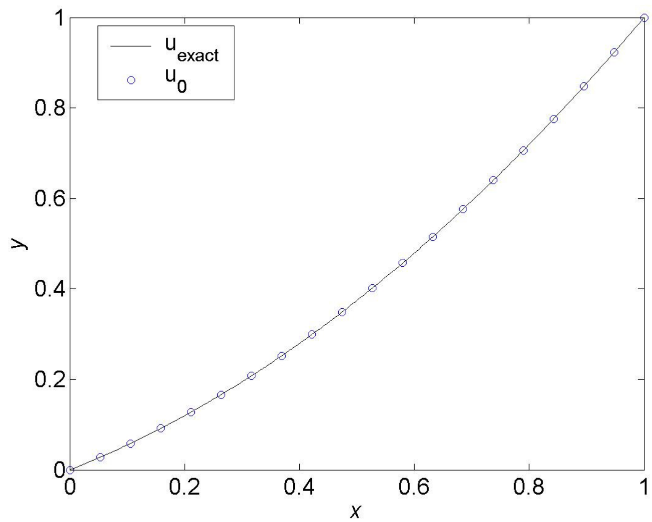

Figure 2.1: The graphs of both u_{exact} and u_0 for the case $\varepsilon = 0.1$.

which can be also rewritten in a more compact way

$$u_{\text{exact}} = \frac{1}{\varepsilon^2}(\cosh \varepsilon x - 1) + \frac{\sinh \varepsilon x}{\varepsilon^2 \sinh \varepsilon}(1 + \varepsilon^2 - \cosh \varepsilon). \tag{2.17}$$

In Figure 2.1, we have plotted the functions $y = u_{\text{exact}}(x)$ and $y = u_0(x)$ for $\varepsilon = 0.1$. The curves are practically indistinguishable, and the error of approximation is small (see the graph of $|u_{\text{exact}}(x) - u_0(x)|$ plotted in Figure 2.2).

One can be surprised of the quality of the approximation. Indeed, we could expect the difference may be of the order ε^2 or in this particular case 10^{-2}. However, looking closely at the second term in the asymptotic expansion (2.16), one case see that its maximum (without the small parameter ε^2) takes itself small values (near 10^{-2}) within the interval $[0, 1]$ what is incidently comparable with the impact of the small parameter. Moreover, this value is independent of the value of ε. This example shows an importance of the constants (functions) related to the corresponding asymptotic terms. In the case under consideration, this constant was in our favor. However, it is not always the same. Recall that the series may be not converge so the coefficients in the asymptotic expansion should not necessarily decrease with number of terms. An example of this will be given later.

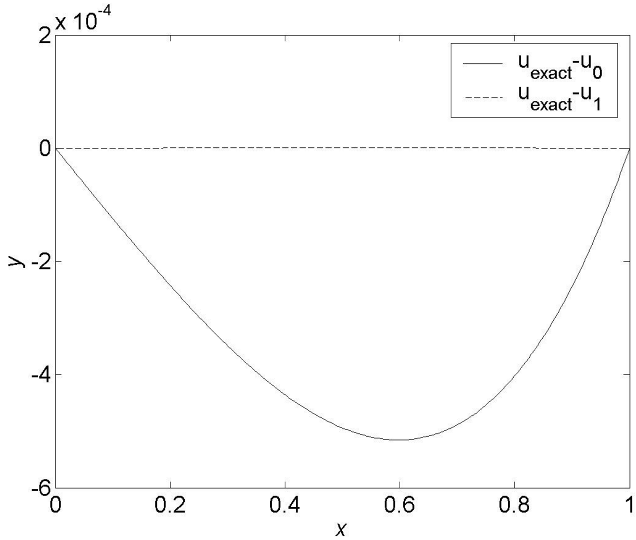

Figure 2.2: Absolute error of approximation for the case $\varepsilon = 0.1$.

Concluding, even the first term of the straightforward asymptotic expansion gives a reasonable approximation to the solution for small values of the parameter ε. If one needs a more accurate approximation, it is always possible to construct next asymptotic terms.

Of course, the quality of asymptotic approximations depends essentially on the value of the small parameter ε itself. To show this, we present four figures (Figs. 2.3−2.6) with the numerical results corresponding to the problem illustrated by Figs. 2.1 and 2.2, but for two greater values $\varepsilon = 0.5$ and $\varepsilon = 1.0$.

However, there could appear special cases when solution to the limit equation $L(u) = 0$ does not exist at all or exist in the space which does not coincide with that one when we are searching a solution from. Other not clear situations may happen when the original problem has more than one solution while using the the straightforward procedure one can find the only one. Usually, all such cases correspond to the so-called *singular asymptotic expansion* or equally to *singular perturbations.* In fact, there is no unique definition for these notions. Other usually used definition says that when the straightforward asymptotic expansion produces a smooth solution of a problem, as a function of small parameter ε, then such a problem is a *regular perturbation problem.* An asymptotic problem is called a *singular perturba-*

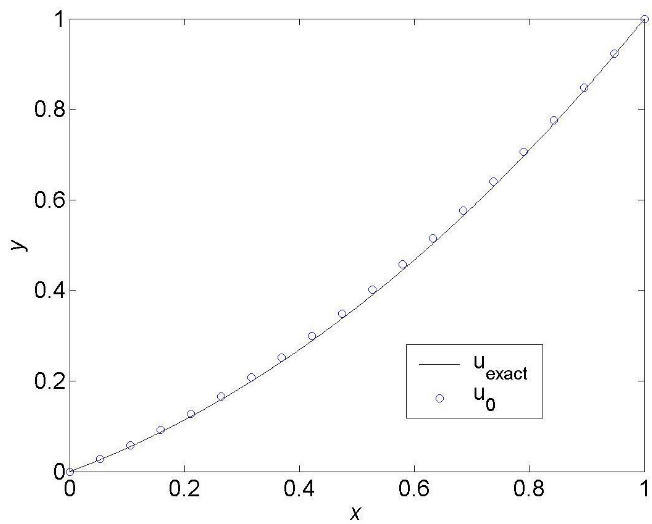

Figure 2.3: The graphs of both u_{exact} and u_0 for the case $\varepsilon = 0.5$.

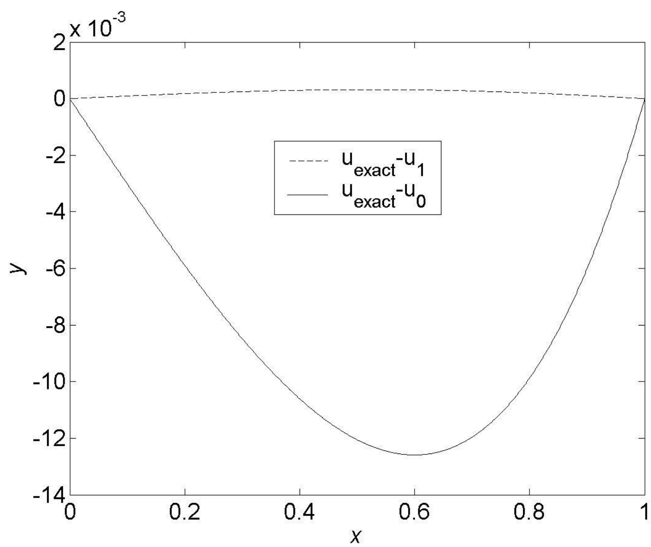

Figure 2.4: Absolute error $u_{\text{exact}} - u_0$ of approximation for the case $\varepsilon = 0.5$.

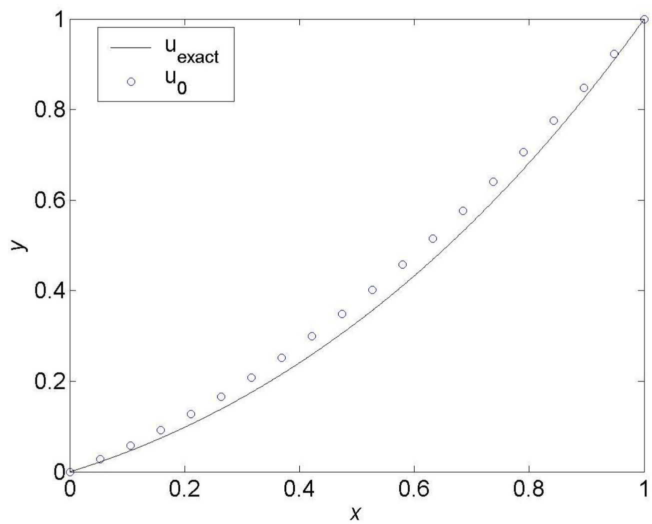

Figure 2.5: The graphs of both u_{exact} and u_0 for the case $\varepsilon = 1.0$.

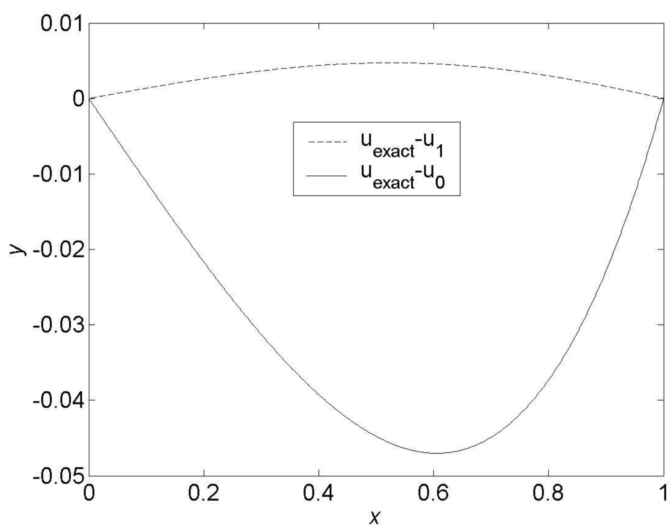

Figure 2.6: Absolute error $u_{\text{exact}} - u_0$ of approximation for the case $\varepsilon = 1.0$.

tion problem otherwise.

However, if we do not know a priori which type of the problems we are dealing with, the straightforward asymptotic expansion is the first necessary step. And only meeting an obstacle and understanding what is its reason, one can move forward to solve the problem with *more advanced asymptotic techniques*. Below we show a few simple examples illuminating various (but not all) problems which one may meet using the straightforward asymptotic procedure.

1.2.3 Example 3. Asymptotic solution of a quadratic equation

Let us consider the following quadratic equation with a small parameter ε:

$$u - 1 + \varepsilon u^2 = 0. \tag{2.18}$$

Then, using the introduced above notation, we conclude $L(u) = u - 1$ and $N(u) = u^2$.

The three-term straightforward asymptotic expansion of the solution $u(\varepsilon)$ gives the result

$$u = 1 - \varepsilon + 2\varepsilon^2 + \ldots, \tag{2.19}$$

which is an approximation to the following one of two solutions of Eq. (2.18):

$$u_1(\varepsilon) = \frac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon} = \frac{2}{1 + \sqrt{1 + 4\varepsilon}}. \tag{2.20}$$

Moreover, using the Taylor series

$$\sqrt{1 + z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 \ldots, \tag{2.21}$$

one can show that the three-term expansion (2.19) of the solution (2.18) differs from the exact solution (2.20) on the term of the order ε^3 or even more accurate:

$$|u(\varepsilon) - 1 + \varepsilon - 2\varepsilon^2| < 10\varepsilon^3. \tag{2.22}$$

The inequality (2.22) holds true for any $0 < \varepsilon < 1$. Therefore, the three-term straightforward asymptotic expansion deviates from the exact solution (2.18) by the term which can be estimated as $10\varepsilon^3$ for small ε. For example, if $\varepsilon = 0.1$, the difference between the exact and approximate solutions is less than 0.01. If $\varepsilon = 0.01$, the difference is less than 0.00001. However, as it follows in this case the constant in front of the term of the order ε^3 of the asymptotic expansion is not a small one (compare with the discussion after Figure 2.2 above).

Strictly speaking, there are two *equally important* steps in asymptotic analysis. First step is to construct a formal asymptotic expansion (as we did in (1.14), (2.16) and (2.19)). The second is to prove that this expansion approximates the exact solution with a proper accuracy as $\varepsilon \to 0$ that is usually even more difficult task. In this module, we will deal mostly with the first step, making remarks if necessary about possible estimates.

The second solution of the quadratic equation (2.18), which has not been found from the straightforward asymptotic expansion, is

$$u_2(\varepsilon) = -\frac{1+\sqrt{1+4\varepsilon}}{2\varepsilon}. \tag{2.23}$$

Note that $u_2(\varepsilon) \to -\infty$ as $\varepsilon \to 0$. It is clear that such type of information is crucial for the further analysis and need to be taken into account when searching for the approximate solution in this case.

Finally, note that apart of the fact that the original problem (2.18) has two solutions, the straightforward asymptotic procedure gives an approximation to only one of them. This is an example of singularly perturbed problem.

1.2.4 Example 4. Asymptotic solution of a quadratic equation with a parameter

Consider now another quadratic equation with an additional parameter τ

$$(u-1)(u-\tau) + \varepsilon u = 0. \tag{2.24}$$

Here $L(u) = (u-1)(u-\tau)$ and $N(u) = u$. There are two solutions of the *limit* equation $L(u) = 0$ as follows:

$$u_0^{(1)} = 1, \quad u_0^{(2)} = \tau.$$

The three-term straightforward asymptotic expansions of the corresponding solutions are obtained by using (1.7) as (details will be given later)

$$u^{(1)} = 1 - \frac{\varepsilon}{1-\tau} - \frac{\varepsilon^2\tau}{(1-\tau)^3} + \dots,$$

$$u^{(2)} = \tau + \frac{\varepsilon\tau}{1-\tau} + \frac{\varepsilon^2\tau}{(1-\tau)^3} + \dots .$$

It is seen that the expansions may be not valid where τ is close to unity. Now, the coefficients at the powers of the small parameter ε may even drastically increase with number of asymptotic terms.

Thus, it should be emphasized that straightforward asymptotic expansions may give inaccurate approximation or even be not valid for some values of the problem parameters.

The final example of a perturbed problem in this section will refer to a *singular* perturbation case.

1.2.5 Example 5. Solution of a singularly perturbed boundary value problem

Let u satisfy the equation

$$\varepsilon^2 u''(x) - u(x) = 1, \quad 0 \le x \le 1, \tag{2.25}$$

with the same boundary conditions (2.13) as above.

The solution $u(x)$ can be interpreted as the temperature of a thin rod connecting two large bodies which are maintained at constant temperature as shown in Figure 2.7. The positive coefficient ε^2 denotes the normalised (relatively small) thermal conductivity, and the temperature of the surrounding medium is equal to -1.

Formally, if we set $\varepsilon = 0$ in (2.25) then the asymptotic approximation u_0 of the solution is given by

$$u_0(x) = -1, \quad 0 < x < 1. \tag{2.26}$$

However, in contrast with the previous example (2.14), it does not satisfy the boundary conditions (2.13). This fact indicates that the perturbation is singular. And need to be corrected. The corrected asymptotic approximation takes form

$$u_{\text{asymp}}(x) = -1 + \mathrm{e}^{-x/\varepsilon} + 2\mathrm{e}^{-(1-x)/\varepsilon}, \quad 0 < x < 1, \tag{2.27}$$

which satisfies Eq. (2.25) and leaves an exponentially small error with respect to the small parameter ε in the boundary conditions (2.13). Two last

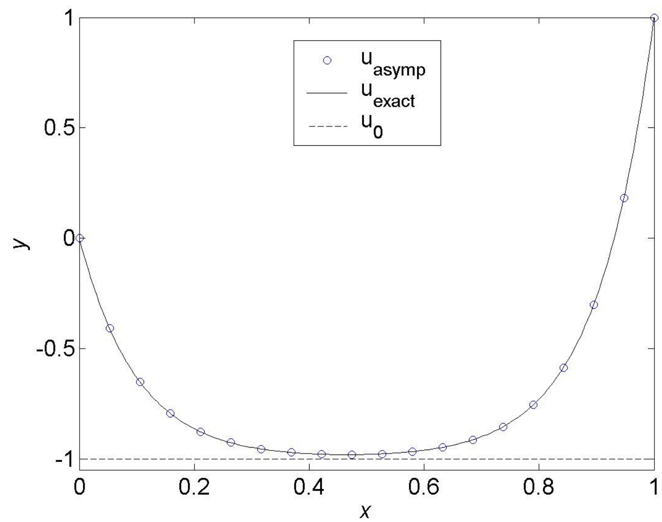

Figure 2.7: A thin rod connecting two bodies of given temperature ($\varepsilon = 0.1$).

correction terms in (2.27) are concentrated near the ends of the interval $(0, 1)$ and are called the *boundary layers*.

In this case, the exact solution of the problem (2.25), (2.13) is

$$u_{\text{exact}}(x) = -1 + \frac{\mathrm{e}^{1/\varepsilon}(\mathrm{e}^{1/\varepsilon} - 2)}{\mathrm{e}^{2/\varepsilon} - 1}\mathrm{e}^{-x/\varepsilon} + \frac{2\mathrm{e}^{1/\varepsilon} - 1}{\mathrm{e}^{2/\varepsilon} - 1}\mathrm{e}^{x/\varepsilon}, \tag{2.28}$$

or in a form, which is better for computation,

$$u_{\text{exact}}(x) = -1 + \frac{(1 - 2\mathrm{e}^{-1/\varepsilon})}{1 - \mathrm{e}^{-2/\varepsilon}}\mathrm{e}^{-x/\varepsilon} + \frac{2 - \mathrm{e}^{-1/\varepsilon}}{1 - \mathrm{e}^{-2/\varepsilon}}\mathrm{e}^{(x-1)/\varepsilon}. \tag{2.29}$$

In Fig. 2.7 we plot the function $y = u_{\text{exact}}(x)$ for the case when $\varepsilon = 0.1$. It can be observed that the quantities u_{exact} and u_0 are quite close to each other in the middle region of the interval $(0, 1)$. However, $|u_{\text{exact}} - u_0|$ becomes large when we approach the end points $x = 0$ and $x = 1$ (see Fig. 2.8). On the other hand, the functions u_{exact} and u_{asymp} are very close, so that it is hardly possible to distinguish between their graphs. In particular, when $\varepsilon = 0.1$, the difference $u_{\text{exact}} - u_{\text{asymp}}$ has the order 10^{-4}.

We note that the asymptotic approximation involving the boundary layer terms can be constructed for the solution of the problem (2.25), (2.13) with-

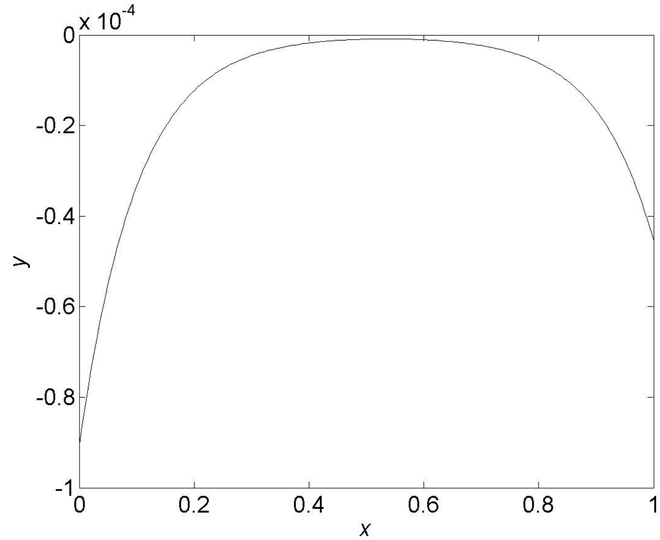

Figure 2.8: Absolute error $u_{\mathrm{exact}} - u_{\mathrm{asymp}}$ of the asymptotic approximation for the case $\varepsilon = 0.1$.

out any knowledge of u_{exact}. That is, in order to compensate for the discrepancy left by u_0 in the boundary condition at $x = 0$, we make the change of the independent variable $X = x/\varepsilon$, and consider a new unknown variable

$$U(X) = u(x) - u_0(x) = u(\varepsilon X) + 1. \tag{2.30}$$

as if we have been looking at a neighbourhood of $x = 0$ through a microscope. The point $x = 1$ moves then to the right through a large distance $1/\varepsilon$, and we can forget about it for a moment assuming it is in infinity. Now we obtain the model problem (independent of the small parameter):

$$\frac{d^2U(X)}{dX^2} - U(X) = 0, \quad X > 0, \tag{2.31}$$

with the boundary conditions

$$U(0) = 1; \quad U(X) \to 0, \quad X \to \infty, \tag{2.32}$$

which has the solution $U(X) = \mathrm{e}^{-X} = \mathrm{e}^{-x/\varepsilon}$. This gives the second term in the asymptotic representation (2.27). In a similar way, we introduce the

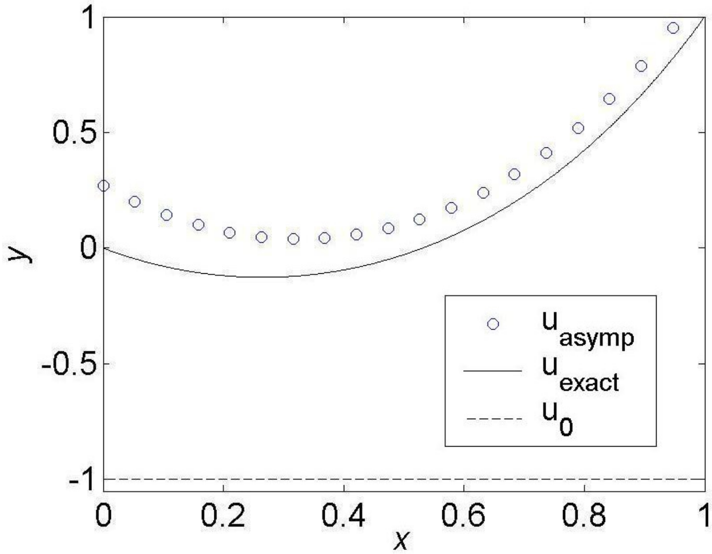

Figure 2.9: A thin rod connecting two bodies of given temperature ($\varepsilon = 0.5$).

scaled variable $Y = (1 - x)/\varepsilon$ and compensate for the discrepancy of u_0 at $x = 1$ by solving the second model problem

$$\frac{d^2V(Y)}{dY^2} - V(Y) = 0, \quad Y > 0 \tag{2.33}$$

$$V(0) = 2; \qquad V(Y) \to 0, \quad Y \to \infty, \tag{2.34}$$

which has the solution $V(Y) = 2\mathrm{e}^{-Y} = 2\mathrm{e}^{(x-1)/\varepsilon}$. The required asymptotic approximation in this case is given by the formula

$$u_{\text{asymp}}(x) = u_0(x) + U(X) + V(Y),$$

which is, in fact, the same as (2.27). This is a simple example of the so-called *multi-scaled compound asymptotic approximation* involving the boundary layer terms $U(X)$ and $V(Y)$. The above argument is an important ingredient of the *method of compound asymptotic expansions* which we are going discuss among others in the next sections.

Finally, in order to illustrate how the quality of asymptotic approximations depends on the value of the small parameter ε, we present Fig. 2.9 with the numerical results corresponding to the problem illustrated by Figs. 2.7 and 2.8, but for a greater value $\varepsilon = 0.5$.

1.3 Preliminary results and main definitions

1.3.1 Landau symbols and gauge functions

Approximate (asymptotic) solution not necessary has the form of a power series as it happened in the previous section. Asymptotic expansions may require more complex functions of the small parameter ε, such as $\log^{-1}\varepsilon$, $e^{-\frac{1}{\varepsilon}}$ and many others. We need these functions to compare the behavior of our solution with them as $\varepsilon \to 0$. Note that there is no problem to compare two smooth real functions, say, f and g taking different (nonzero) values at a specific point $x = a$. Indeed, if $f(a) > g(a)$, then there exists an interval $(a - \delta, a + \delta)$ where $f(x) > g(x)$ for any $x \in (a - \delta, a + \delta)$. However, if $f(a) = g(a)$ nothing can be said in advance. Without lost of generality, we can consider in this case that $f(a) = g(a) = 0$. We now will try to define how to compare the functions in this particular case and what is the meaning of that "comparison".

Consider a function $f(\varepsilon)$ of a single real parameter ε. We are interested in the behavior of this function as ε tends to zero. This behavior might depend on whether ε tends to zero from below, denoted as $\varepsilon \to -0$, or from above, denoted as $\varepsilon \to +0$. For example,

$$\lim_{\varepsilon \to -0} e^{-\frac{1}{\varepsilon}} = +\infty, \quad \lim_{\varepsilon \to +0} e^{-\frac{1}{\varepsilon}} = 0.$$

In order to fix our ideas, we always assume in the sequel that

$$\varepsilon \geq 0. \tag{3.1}$$

Definition 1. We say that a function $f : (0, a) \to \mathbb{R}$ is an infinitesimally small one at the point $\varepsilon = 0$ if

$$\lim_{\varepsilon \to +0} f(\varepsilon) = 0. \tag{3.2}$$

For an example, function $f(x) = x^2$ is an infinitesimally small one at the point $x = 0$. However, it is clear that the statement is wrong for the point $x = 1$ as well as any other point $x \neq 0$!

Information given by the limit (3.2) is useful but such limits do not explain how quickly the function tends to this limit. In applied mathematics, we need more detailed information about the limiting behaviour of $f(\varepsilon)$ as $\varepsilon \to 0$.

Definition 2. Let $f, g : (0, a) \to \mathbb{R}$ be real functions. If there exist positive constants C and δ such that

$$|f(\varepsilon)| \leq C|g(\varepsilon)|, \quad \text{for any} \quad 0 < \varepsilon < \delta, \tag{3.3}$$

then we say that $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to 0$ (*function f is a Big-Oh of the function g at zero point*).

Remark 1. The fact that $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to 0$ does not necessarily imply that $g(\varepsilon) = O(f(\varepsilon))$ at the same point. This only takes place if there exist positive constants $C_1 < C_2$ and δ such that

$$C_1|g(\varepsilon)| \leq |f(\varepsilon)| \leq C_2|g(\varepsilon)|, \quad \text{for any} \quad 0 < \varepsilon < \delta. \tag{3.4}$$

Definition 2 can written in an equivalent form if additionally one assumes that f, g are different from zero in an open interval $(0, \delta)$.

Definition 2a. Let $f, g : (0, a) \to \mathbb{R}$ be real nonzero functions. Then $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to 0$ if

$$\left|\frac{f(\varepsilon)}{g(\varepsilon)}\right| \leq C, \quad \text{for any} \quad 0 < \varepsilon < \delta. \tag{3.5}$$

In other words, the symbol *Big-Oh* in the notation $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to 0$ indicates that the function f can be effectively compared in value with the function g at point $\varepsilon = 0$. Note that to say functions f, g are comparable without indicating at which point makes NO sense.

Imagine now that instead of the previous Definition 2a we have found that

Definition 3. Let $f, g : (0, a) \to \mathbb{R}$ be real nonzero functions and for any small c there exists $\delta = \delta(c)$ such that

$$\left|\frac{f(\varepsilon)}{g(\varepsilon)}\right| \leq c, \quad \text{for any} \quad 0 < \varepsilon < \delta. \tag{3.6}$$

In this case we say $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \to 0$ (*function f is a little-oh of the function g at zero point*).

It is interesting to note that the Definition 3 completely coincides with the following one.

Definition 3a. Let $f, g : (0, a) \to \mathbb{R}$ be real nonzero functions and

$$\lim_{\varepsilon \to +0} \frac{f(\varepsilon)}{g(\varepsilon)} = 0. \tag{3.7}$$

Then, we say that $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \to 0$.

Remark 2. In this case other notation is often used: $f \ll g$ as $\varepsilon \to 0$.

Remark 3. Using Definition 3a one can say that function $f(\varepsilon)$ is an infinitesimally small at point $\varepsilon = 0$ if and only if $f(\varepsilon) = o(1)$ as $\varepsilon \to 0$ or equivalently if $f(\varepsilon) \ll 1$ as $\varepsilon \to 0$.

It is interesting to answer the question: is it possible to rewrite Definition 3a in an equivalent way using *limit* notation? The answer is positive.

Proposition 1. If f, g are real valued functions defined in the interval $(0, a)$ and $g(\varepsilon) \neq 0$ then $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to 0$, if and only if

$$\limsup_{\varepsilon \to +0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| < \infty. \tag{3.8}$$

If instead of lim sup the standard lim will be used in (3.6) then the statement becomes more strong and the following statement can be proved.

Proposition 2. If f, g are real valued functions defined in the interval $(0, a)$, $g(\varepsilon) \neq 0$ and there exists a nonzero constant C such that

$$\lim_{\varepsilon \to +0} \frac{f(\varepsilon)}{g(\varepsilon)} = C, \tag{3.9}$$

then $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to 0$ and $g(\varepsilon) = O(f(\varepsilon))$ as $\varepsilon \to 0$, simultaneously.

Although, definitions given by conditions (3.9) and (3.8) (or (3.5)) are not equivalent to each other, the first one is often more useful and sometimes considered in text books as the main one. Practically the following definition explains this.

Definition 4. If f and g are real valued functions defined in the interval $(0, a)$, $g(\varepsilon) \neq 0$ and

$$\lim_{\varepsilon \to +0} \frac{f(\varepsilon)}{g(\varepsilon)} = 1, \tag{3.10}$$

then we say function $f(\varepsilon)$ is equivalent to the function $g(\varepsilon)$ as $\varepsilon \to 0$ or $f(\varepsilon) \sim g(\varepsilon)$ as $\varepsilon \to 0$.

A few examples are presented below. As $\varepsilon \to 0$, the following asymptotic formulas hold true:

$$\sin(\varepsilon) = O(\varepsilon), \qquad \sin(\varepsilon) = o(1), \qquad \sin(7\varepsilon) = O(\varepsilon), \qquad \sin(\varepsilon^4) = O(\varepsilon^4),$$

$$e^{-1/\varepsilon} = o(\varepsilon^n) \ \ \text{(for any } n\text{)}, \qquad \frac{1}{1-\varepsilon} = 1 + O(\varepsilon), \qquad 1 - \cos\varepsilon = o(\varepsilon),$$

$$\frac{1}{1-\varepsilon} = O(\varepsilon^{-\frac{1}{2}}), \qquad \sinh\varepsilon = O(\varepsilon), \qquad \cosh\varepsilon - 1 = O(\varepsilon^2),$$

$$\sin\varepsilon \sim \varepsilon, \qquad \sin(7\varepsilon) \sim 7\varepsilon, \qquad \sin\varepsilon + \log\left(\frac{1}{\varepsilon}\right) \sim \log\left(\frac{1}{\varepsilon}\right).$$

Some of them are straightforward, other need more accurate proof. The following refreshment from the Calculus module will be of help.

Example 1. Let us consider function $f(\varepsilon) = \sin\varepsilon$. Taylor expansion of $\sin\varepsilon$ gives the following result:

$$\sin\varepsilon = \sum_{k=0}^{N} (-1)^k \frac{\varepsilon^{2k+1}}{(2k+1)!} + R_N(\varepsilon). \tag{3.11}$$

Here the remainder term can be estimated as follows:

$$R_N(\varepsilon) = \frac{\varepsilon^{2N+2}}{(2N+2)!}\theta(\varepsilon),$$

where $|\theta(\varepsilon)| \leq 1$ is an unknown function and, according to the definition (3.3),

$$R_N(\varepsilon) = O(\varepsilon^{2N+2}), \quad \text{as} \quad \varepsilon \to 0. \tag{3.12}$$

Note that the respective Taylor series

$$\sin\varepsilon = \sum_{k=0}^{\infty} (-1)^k \frac{\varepsilon^{2k+1}}{(2k+1)!}, \tag{3.13}$$

converges for any finite value of ε.

Writing down a few first terms we have

$$\sin\varepsilon = \varepsilon - \frac{1}{3!}\varepsilon^3 + \frac{1}{5!}\varepsilon^5 + \ldots = \varepsilon - \frac{1}{6}\varepsilon^3 + \frac{1}{120}\varepsilon^5 + \ldots.$$

Therefore, $\sin\varepsilon$ tends to zero similarly to ε^1 as $\varepsilon \to 0$, therefore $\sin\varepsilon = O(\varepsilon)$ as $\varepsilon \to 0$, or more accurate:

$$\sin\varepsilon \sim \varepsilon, \quad \text{as} \quad \varepsilon \to 0. \tag{3.14}$$

In the same way, as $\varepsilon \to 0$, we find

$$\sin\varepsilon - \varepsilon = O(\varepsilon^3), \quad \sin\varepsilon - \varepsilon + \frac{1}{6}\varepsilon^3 = O(\varepsilon^5).$$

One can prove in the general case

$$\sin\varepsilon - \sum_{k=0}^{N}(-1)^k\frac{\varepsilon^{2k+1}}{(2k+1)!} = O(\varepsilon^{2N+3}), \quad \varepsilon\to 0, \tag{3.15}$$

Here we used the fact that the terms change their signs with the number N. The latter can be written in an equivalent form as follows:

$$\sin\varepsilon = \sum_{k=0}^{N}(-1)^k\frac{\varepsilon^{2k+1}}{(2k+1)!} + O(\varepsilon^{2N+3}) \quad \text{as} \quad \varepsilon\to 0. \tag{3.16}$$

Note that this result is stronger than that discussed in (3.11) and (3.12). For reason which will be clear later, we also rewrite (3.16) the following (weaker) manner:

$$\sin\varepsilon = \sum_{k=0}^{N}(-1)^k\frac{\varepsilon^{2k+1}}{(2k+1)!} + o(\varepsilon^{2N+1}) \quad \text{as} \quad \varepsilon\to 0. \tag{3.17}$$

Example 2. Consider other trigonometric function $f(\varepsilon)=\cos\varepsilon$ with its Taylor expansion

$$\cos\varepsilon = \sum_{k=0}^{\infty}(-1)^k\frac{\varepsilon^{2k+2}}{(2k+2)!}, \tag{3.18}$$

which converges for an arbitrary $\varepsilon\in\mathbb{R}$. Then the same arguments as above give $1-\cos\varepsilon = o(\varepsilon)$ as $\varepsilon\to 0$,

$$1-\cos\varepsilon \sim \frac{1}{2}\varepsilon^2 \quad \text{as} \quad \varepsilon\to 0, \tag{3.19}$$

and

$$\cos\varepsilon = \sum_{k=0}^{N}(-1)^k\frac{\varepsilon^{2k}}{(2k)!} + O(\varepsilon^{2N+2}) \quad \text{as} \quad \varepsilon\to 0. \tag{3.20}$$

Note however, that as $x\to\pi/2$, we will have

$$\sin(x) = O(1), \quad \cos(x) = o(1),$$

that is completely different in comparison with (3.14) and (3.19)

Below we present useful relationships immediately following from Taylor expansions of some elementary functions:

$$(1+\varepsilon)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} \varepsilon^k, \tag{3.21}$$

where the series converges for all $|\varepsilon| < 1$ while the binomial coefficients are given by the formula

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!}$$

In the particular case $\alpha = -1$, we have the so-called geometrical series

$$\frac{1}{1+\varepsilon} = \sum_{k=0}^{\infty} (-1)^k \varepsilon^k. \tag{3.22}$$

Other useful Taylor expansions are:

$$\ln(1+\varepsilon) = \sum_{k=0}^{\infty} (-1)^k \frac{\varepsilon^{k+1}}{k+1}, \tag{3.23}$$

$$e^\varepsilon = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!}, \tag{3.24}$$

$$\sinh \varepsilon = \sum_{k=0}^{\infty} \frac{\varepsilon^{2k+1}}{(2k+1)!}, \tag{3.25}$$

$$\cosh \varepsilon = \sum_{k=0}^{\infty} \frac{\varepsilon^{2k}}{(2k)!}. \tag{3.26}$$

Moreover, the expansion (3.23) converges for $\varepsilon \in (-1, 1]$ while others for any $\varepsilon \in \mathbb{R}$.

Additionally to relationships (3.14) and (3.19) the following simple ones can be directly obtained from (3.21) – (3.24):

$$(1+\varepsilon)^\alpha - 1 \sim \alpha\varepsilon \quad \text{as} \quad \varepsilon \to 0, \tag{3.27}$$

$$\tan \varepsilon \sim \varepsilon \quad \text{as} \quad \varepsilon \to 0, \tag{3.28}$$

$$\sinh\varepsilon \sim \varepsilon \quad \text{as} \quad \varepsilon \to 0, \tag{3.29}$$

$$\cosh\varepsilon - 1 \sim \frac{1}{2}\varepsilon^2 \quad \text{as} \quad \varepsilon \to 0, \tag{3.30}$$

$$\tanh\varepsilon \sim \varepsilon \quad \text{as} \quad \varepsilon \to 0, \tag{3.31}$$

$$x^\varepsilon - 1 \sim \varepsilon \ln x \quad \text{as} \quad \varepsilon \to 0, \tag{3.32}$$

$$\log_x \varepsilon - 1 \sim \frac{\varepsilon}{\ln x} \quad \text{as} \quad \varepsilon \to 0, \tag{3.33}$$

where $0 < x$ is an arbitrary value such that $x \neq 1$.

In the aforementioned examples, we compare the given functions with known functions called *gauge functions*. The gauge functions $g(\varepsilon)$ are the functions with well understood behaviour as $\varepsilon \to 0$. We will use them to describe the behaviour of a function $f(\varepsilon)$ under consideration for small ε in terms of the introduced special symbols of order: O (Big-Oh), o (little-oh) and $\sim$ (equivalent).

The most useful of them are power gauge functions

$$\ldots, \varepsilon^{-n}, \ldots, \varepsilon^{-2}, \quad \varepsilon^{-1}, \quad 1, \quad \varepsilon, \quad \varepsilon^2, \ldots, \varepsilon^n, \ldots .$$

Note that sometimes it also makes sense to describe bahaviour of unknown functions in terms of other elementary functions, for instance,

$$\tan\varepsilon, \quad \cos\varepsilon, \quad \sin\varepsilon, \quad \sinh\varepsilon, \ldots .$$

with well-known behaviour (see relationships (3.22) – (3.33).

However, in some cases we need more complex gauge functions as for example for any natural value k:

$$\sin\left(\frac{1}{\varepsilon}\right), \quad \sqrt[k]{\varepsilon}, \quad \log^{-k}\left(\frac{1}{\varepsilon}\right), \quad \varepsilon\log^{k}\left(\frac{1}{\varepsilon}\right), \quad \mathrm{e}^{-\frac{k}{\varepsilon}}, \quad \log^{-k}\left(\log\left(\frac{1}{\varepsilon}\right)\right), \ldots,$$

as they cannot be represented by the power gauge functions. One can check that

$$\sin\left(\frac{1}{\varepsilon}\right) = O(1), \quad \text{as} \quad \varepsilon \to 0,$$

while the estimate $1 = O(\sin\left(\frac{1}{\varepsilon}\right))$ as $\varepsilon \to 0$ is not true. Indeed, there is a sequence $\varepsilon_n = 1/(\pi n)$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and $\sin(1/\varepsilon_n) = 0$.

Others gauge functions are of essentially different orders $o(1)$ where *the smallest one* is the function $\mathrm{e}^{-\frac{k}{\varepsilon}}$. For example

$$\mathrm{e}^{-\frac{k}{\varepsilon}} = o\left(\sqrt[m]{\varepsilon}\right), \quad \text{as} \quad \varepsilon \to 0,$$

for arbitrary natural k, m.

If function $f(x, \varepsilon)$ depends on two variables $\varepsilon \in (0, a)$ as above and additionally on another variable $x \in X$ (see for example (3.32) and (3.33)), and $g(x, \varepsilon)$ is a gauge function, we also write

$$f(x, \varepsilon) = O\left(g(x, \varepsilon)\right) \;\; \text{as} \;\; \varepsilon \to 0 \tag{3.34}$$

for any fixed value of $x \in X$.

According the definition this means that for any fixed value x there exist constants C and δ both independent of ε (generally speaking, $C = C(x)$ and $\delta = \delta(x)$) such that the following inequality holds true:

$$|f(x, \varepsilon)| \leq C|g(x, \varepsilon)| \quad \text{for} \quad 0 < \varepsilon < \delta. \tag{3.35}$$

If the constants C and δ can be chosen independently of x (in a way that inequality (3.35) is valid for all values $x \in X$) then we say that the relation asymptotic estimate (3.35)) holds true *uniformly.*

Definition 5. By using the limit-definitions from (3.7), (3.9) and (3.9) where we replace the limit with

$$\lim_{\varepsilon \to 0} \left| \frac{f(x, \varepsilon)}{g(x, \varepsilon)} \right|,$$

we say that

$f(x, \varepsilon) = O[g(x, \varepsilon)]$ as $\varepsilon \to 0$ uniformly wrt (with respect to) x from the interval (a, b) if the limit in (3.7) is finite for any x from this interval;

$f(x, \varepsilon) = o[g(x, \varepsilon)]$ as $\varepsilon \to 0$ uniformly wrt x from the interval (a, b) if the limit in (3.9) is zero for any x from this interval;

$f(x, \varepsilon) \sim g(x, \varepsilon)$ as $\varepsilon \to 0$ uniformly wrt x from the interval (a, b) if the limit in (3.10) is equal to one for any x from this interval.

Examples

a) $(x + \varepsilon)^2 - x^2 = O(\varepsilon)$ nonuniformly as $\varepsilon \to 0$, for $x \in \mathbb{R}$,

b) $(x + \varepsilon)^2 - x^2 = O(\varepsilon)$ uniformly as $\varepsilon \to 0$, for $|x| < 10$,

c) $\varepsilon = O\left((x+\varepsilon)^2 - x^2\right)$ nonuniformly as $\varepsilon \to 0$, for $x \in (0, 10)$,
d) $\varepsilon = O\left((x+\varepsilon)^2 - x^2\right)$ uniformly as $\varepsilon \to 0$, for $|x| > 10$,
e) $\sin(x+\varepsilon) = O(1)$ uniformly as $\varepsilon \to 0$, for $|x| \in \mathbb{R}$,
f) $\sqrt{x+\varepsilon} - \sqrt{x} = O(\varepsilon)$ nonuniformly as $\varepsilon \to 0$, for $x \in \mathbb{R}_+$,
g) $\sqrt{x+\varepsilon} - \sqrt{x} = O(\varepsilon)$ uniformly as $\varepsilon \to 0$, for $x \in (1, 2)$,
h) $e^{-\varepsilon x} - 1 = O(\varepsilon)$ nonuniformly as $\varepsilon \to 0$, for $x \in \mathbb{R}_+$,
i) $e^{-\varepsilon x} - 1 = O(\varepsilon)$ nonuniformly as $\varepsilon \to 0$, for $x \in (0, 1)$,
j) $\varepsilon = O\left(e^{-\varepsilon x} - 1\right)$ uniformly as $\varepsilon \to 0$, for $x \in (1, 2)$.

Relations written in the forms

$$f(x,\varepsilon) = O[g(x,\varepsilon)], \quad f(x,\varepsilon) = o[g(x,\varepsilon)], \quad f(x,\varepsilon) \sim g(x,\varepsilon) \quad \text{as } \varepsilon \to 0$$

are known as *asymptotic formulas* or *asymptotic estimates*.

Theorem 1. The following formulas hold true:

$$o[f(x,\varepsilon)] + o[f(x,\varepsilon)] = o[f(x,\varepsilon)] \quad \text{as} \quad \varepsilon \to 0, \tag{3.36}$$

$$o[f(x,\varepsilon)] \cdot o[g(x,\varepsilon)] = o[f(x,\varepsilon) \cdot g(x,\varepsilon)] \quad \text{as} \quad \varepsilon \to 0, \tag{3.37}$$

$$o\{o[f(x,\varepsilon)]\} = o[f(x,\varepsilon)] \quad \text{as} \quad \varepsilon \to 0, \tag{3.38}$$

$$O[f(x,\varepsilon)] + O[f(x,\varepsilon)] = O[f(x,\varepsilon)] \quad \text{as} \quad \varepsilon \to 0, \tag{3.39}$$

$$O[f(x,\varepsilon)] \cdot O[g(x,\varepsilon)] = O[f(x,\varepsilon) \cdot g(x,\varepsilon)] \quad \text{as} \quad \varepsilon \to 0, \tag{3.40}$$

$$O\{O[f(x,\varepsilon)]\} = O[f(x,\varepsilon)] \quad \text{as} \quad \varepsilon \to 0, \tag{3.41}$$

$$o[f(x,\varepsilon)] + O[f(x,\varepsilon)] = O[f(x,\varepsilon)] \quad \text{as} \quad \varepsilon \to 0, \tag{3.42}$$

$$o[f(x,\varepsilon)] \cdot O[g(x,\varepsilon)] = o[f(x,\varepsilon) \cdot g(x,\varepsilon)] \quad \text{as} \quad \varepsilon \to 0, \tag{3.43}$$

$$O\{o[f(x,\varepsilon)]\} = o[f(x,\varepsilon)] \quad \text{as} \quad \varepsilon \to 0, \tag{3.44}$$

$$o\{O[f(x,\varepsilon)]\} = o[f(x,\varepsilon)] \quad \text{as} \quad \varepsilon \to 0, \tag{3.45}$$

All formulas (3.36) – (3.45) can be easy verified by the limit-definitions. However, to prove that an estimate is uniform, one should accurately check it according to the definition in every particular case or to ensure that all the fractional expressions exhibit this property.

Theorem 2. Asymptotic formulas can be integrated with respect to the small parameter ε.

For example if $f(x,\varepsilon) = o[g(x,\varepsilon)]$ as $\varepsilon \to 0$ then

$$\int_0^\varepsilon f(x,\varepsilon)d\varepsilon = o\left[\int_0^\varepsilon g(x,\varepsilon)d\varepsilon\right] \quad \text{as} \quad \varepsilon \to 0. \tag{3.46}$$

It is proved by using L′Hospital's rule.

Example 4. Abelian and Tauberian type theorems. First let us remind that integration remains true after proper integration.

Theorem 3. Let f be integrable function on the positive real axis ($f \in L_{loc}(R^+)$) and for some $\alpha > 0$ the following asymptotic equivalence takes place:

$$f(t) \sim t^\alpha \quad \text{as} \quad t \to \infty,$$

then

$$F(t) \equiv \int_0^t f(x)dx \sim (\alpha+1)^{-1}t^{\alpha+1} \quad \text{as} \quad t \to \infty.$$

The proof is immediate by using L′Hospital rule. Such statements are usually known as Abelian type theorems. Inverse theorems are known as Tauber theorems. Example of Tauberian type theorem is given below.

Theorem 4. Let f be a monotonic (increasing) function $f \in L_{loc}(R^+)$ and for some $\alpha > 0$ the following estimate takes place:

$$F(t) = \int_0^t f(x)dx \sim (\alpha+1)^{-1}t^{\alpha+1} \quad \text{as} \quad t \to \infty,$$

then

$$f(t) \sim t^\alpha \quad \text{as} \quad t \to \infty.$$

From the first glance it looks like the Theorems are equivalent (if and only if conditions). However, it is not true. In the second one much stronger condition is required (monotonicity). Explanation for this is quite clear.

The first theorem results in integration of the asymptotic expansion while the second one is the inverse theorem (formal differentiation).

Remark 4. Asymptotic formulas cannot be differentiated, in general. As an example consider the true estimate

$$\varepsilon \sin\left(\frac{1}{\varepsilon^3}\right) = o\left(\varepsilon^{-\frac{1}{2}}\right), \quad \text{as} \quad \varepsilon \to 0,$$

Differentiating both sides results in a new wrong relationship:

$$\sin(\varepsilon^{-3}) - 3\varepsilon^{-3}\cos(\varepsilon^{-3}) = o\left(-\frac{1}{2}\varepsilon^{-\frac{3}{2}}\right) \equiv o(\varepsilon^{-\frac{3}{2}}) \quad \text{as} \quad \varepsilon \to 0.$$

1.3.2 Asymptotic sequences and asymptotic expansions

A sequence $\delta_n(\varepsilon)$, $n = 1, 2, \ldots$, of functions of ε is called an *asymptotic sequence* if

$$\delta_{n+1}(\varepsilon) = o(\delta_n(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0, \tag{3.1}$$

for each $n = 1, 2, \ldots$. Here $\delta_n(\varepsilon)$ are the gauge functions. If the $\delta_n(x, \varepsilon)$ depend on a variable $x \in X$, one can consider also uniformity with respect to the additional variable x.

Examples (everywhere $\varepsilon \to 0$):

a) $\delta_n(\varepsilon) = \varepsilon^n$,

b) $\delta_n(\varepsilon) = \varepsilon^{\lambda_n}, \qquad (\lambda_{n+1} > \lambda_n)$,

c) $\delta_0(\varepsilon) = \log\varepsilon$, $\delta_1(\varepsilon) = 1$, $\delta_2(\varepsilon) = \varepsilon\log\varepsilon$, $\delta_3(\varepsilon) = \varepsilon$, $\delta_4(\varepsilon) = \varepsilon^2\log^2\varepsilon$, $\delta_5(\varepsilon) = \varepsilon^2\log\varepsilon$, $\delta_6(\varepsilon) = \varepsilon^2$, $\ldots$.

Definition 6. A sum of the form

$$\sum_{n=1}^{N} a_n(x)\delta_n(\varepsilon) \tag{3.2}$$

is called an *asymptotic expansion* of the function $f(x, \varepsilon)$ as $\varepsilon \to 0$ up to N asymptotic terms with respect to the asymptotic sequence $\delta_n(\varepsilon)$ if

$$f(x, \varepsilon) - \sum_{n=1}^{M} a_n(x)\delta_n(\varepsilon) = o(\delta_M(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0 \tag{3.3}$$

for each $M = 1, 2, \ldots, N$. Thus the reminder is always smaller than the last term included in the asymptotic expansion once ε is sufficiently small.

In case $N = \infty$ we write:

$$f(x,\varepsilon) \sim \sum_{n=1}^{\infty} a_n(x)\delta_n(\varepsilon) \quad \text{as} \quad \varepsilon \to 0. \tag{3.4}$$

Definition 7. Asymptotic expansion (3.3) is said to be *uniformly valid* in some domain D ($x \in X$) if the consequent asymptotic formulas ($1 \leq M \leq N$) hold true *uniformly*.

For a given asymptotic sequence $\delta_n(\varepsilon)$ and a given function $f(x,\varepsilon)$ the coefficients $a_n(x)$ are uniquely defined. Indeed, let consider the first relationship ($M = 1$):

$$f(x,\varepsilon) - a_1(x)\delta_1(\varepsilon) = o(\delta_1(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0,$$

or equivalently

$$\lim_{\varepsilon \to 0} \frac{f(x,\varepsilon) - a_1(x)\delta_1(\varepsilon)}{\delta_1(\varepsilon)} = 0,$$

that finally gives

$$a_1(x) = \lim_{\varepsilon \to 0} \frac{f(x,\varepsilon)}{\delta_1(\varepsilon)}.$$

For the next step we have

$$f(x,\varepsilon) - a_1(x)\delta_1(\varepsilon) - a_2(x)\delta_2(\varepsilon) = o(\delta_2(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0,$$

or finally

$$a_2(x) = \lim_{\varepsilon \to 0} \frac{f(x,\varepsilon) - a_1(x)\delta_1(\varepsilon)}{\delta_2(\varepsilon)}.$$

The sequence can be continued to any number $M \leq N$.

Remark 5. Note that asymptotic expansion (3.3) should not be represented by a converging series as there is no another condition for it apart from the condition that it has to be constructed on a asymptotic sequence (3.1). However, it may happen that a particular asymptotic serious also converges in some region. As a simple example one can check that asymptotic expansion (3.17) leads to a converging serious for an arbitrary $\varepsilon \in \mathbb{R}$ (now it is clear the reason for this special form of the Taylor expansion (3.16) was chosen). Naturally, the same conclusion can be drawn for any of the relationships (3.20) – (3.26).

Example 4. Estimation of Stieltjes function. Let us consider the so-called Stieltjes function defined by the following integral:

$$S(\varepsilon) = \int_0^\infty \frac{e^{-t}\,dt}{1+\varepsilon t}. \tag{3.5}$$

It is clear that $S(0) = 1$, $S(\varepsilon)$ is a smooth function ($S \in C^\infty(\mathbb{R}_+)$) and it decreases such that

$$\lim_{\varepsilon\to+\infty} S(\varepsilon) = 0.$$

Below we estimate in more details the behaviour of the function near point $\varepsilon = 0$.

One can check straightforward the following identity:

$$\frac{1}{1+\varepsilon t} = \sum_{j=0}^{N} (-\varepsilon)^j t^j + (-\varepsilon)^{N+1} \frac{t^{N+1}}{1+\varepsilon t},$$

then expression (3.5) can be written in the following manner:

$$S(\varepsilon) = \sum_{j=0}^{N} (-\varepsilon)^j \int_0^\infty t^j e^{-t}\,dt + E_N(\varepsilon), \tag{3.6}$$

where the reminder takes the form

$$E_N(\varepsilon) = (-\varepsilon)^{N+1} \int_0^\infty \frac{t^{N+1} e^{-t}\,dt}{1+\varepsilon t}.$$

Note that the first integral in (3.6) is the definition of the Euler Gamma function $\Gamma(x)$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}, \tag{3.7}$$

and when $x = j+1 \in \mathbb{N}$ one can compute it $\Gamma(j+1) = j!$ As a result, we can rewrite (3.6)

$$S(\varepsilon) = \sum_{j=0}^{N} (-\varepsilon)^j j! + E_N(\varepsilon), \tag{3.8}$$

where it is still the exact formula.

If the integral $E_N(\varepsilon)$ is neglected, then the summation represents the $N+1$ terms of an asymptotic expansion. The corresponding series is *divirgent.*

Since this is an algebraic series we can compute its radius of convergence to have:

$$\frac{1}{R} = \lim_{j\to\infty} \left| \frac{(-\varepsilon)^{j+1}(j+1)!}{(-\varepsilon)^j j!} \right| = \lim_{j\to\infty} |-(j+1)\varepsilon| = 0, \quad \text{for any} \quad \varepsilon > 0.$$

However, the series is asymptotic as $\varepsilon \to 0$. By definition, we need to prove that $E_n(\varepsilon) = o(\varepsilon^N)$ as $\varepsilon \to 0$ or

$$\lim_{\varepsilon\to 0} \left\{ \varepsilon \int_0^\infty \frac{t^{N+1}e^{-t}}{1+\varepsilon t}\, dt \right\} = 0.$$

for any fixed and finite N. Let us consider separately the integral and show that it is bounded as $\varepsilon \to$. Indeed,

$$\int_0^\infty \frac{t^{N+1}e^{-t}}{1+\varepsilon t}\, dt < \int_0^\infty t^{N+1}e^{-t}\, dt = N!$$

that finishes the proof.

Remark 6. One can think that Remark 5 gives a superiority for the converging series. However, this is not true. To understand this paradox one can compare two series for the same function $f(\varepsilon)$: one is an asymptotic series constructed for $\varepsilon \to 0$ and other the converging Taylor series built at point $\varepsilon = \varepsilon_0 \neq 0$ but valid in a domain containing the point $\varepsilon = 0$. Then, apart of the fact that the first serious does not generally converge, it provides a better approximation to the function $f(\varepsilon)$ near the point $\varepsilon = 0$.

Asymptotic expansions (approximations, representations) can be integrated, added, subtracted, multiplied and divided resulting in the correct asymptotic expansions for the sum, difference, product and quotient (perhaps based on an enlarged asymptotic sequence).

Moreover, one asymptotic expansion can be substituted into another (however, a special care is needed). Let we consider the following example. Let

$$f(x) = e^{x^2}, \quad x = \varepsilon + \frac{1}{\varepsilon},$$

and we seek for an asymptotic representation for the function when $\varepsilon \to 0$. Formally speaking,

$$x \sim \frac{1}{\varepsilon}, \quad x^2 \sim \frac{1}{\varepsilon^2}, \quad \text{and finally} \quad f(x) \sim e^{\frac{1}{\varepsilon^2}}, \quad \varepsilon \to 0,$$

however, the final result is not correct. More accurate analysis gives

$$f(x) = e^{(\varepsilon+\frac{1}{\varepsilon})^2} = e^{\varepsilon^2+2+\varepsilon^{-2}} \sim e^2 \cdot e^{\frac{1}{\varepsilon^2}}, \quad \varepsilon \to 0.$$

Interestingly, the first incorrect result for the asymptotic equivalence gives the correct one in terms of *Big-Oh* where the particular value of the constant C in definition (3.23) is not important.

1.4 Asymptotic analysis of algebraic equations

1.4.1 Asymptotic expansions for roots

Algebraic equations with a small parameter come analysis of eigen-value problems for matrices as well as from various mechanical problems for example from analysis of dynamic systems (stationary points).

Main idea to construct proper asymptotic expansion comes is to balance a few terms involved the equation. Usually some physical arguments lie behind. We explain this in the following example.

Problem 1. Let $\varepsilon > 0$ be a small parameter. Find asymptotic expansions for all four roots of the 4-th order polynomials:

$$\varepsilon^2 x^4 - \varepsilon x^2 - x - 1 = 0. \tag{4.1}$$

Remark 7. Note that the following options are possible: (a) all zeros are real; (b) two zeros are real and two are conjugate complex ones or (c) there are two complex conjugate pairs of zeros.

Straightforward asymptotic expansion in form

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3), \quad \text{as} \quad \varepsilon \to 0, \tag{4.2}$$

gives one root. Indeed, one can easy check that

$$x^2 = x_0^2 + 2\varepsilon x_1 x_0 + \varepsilon^2(x_1^2 + 2x_0 x_2) + O(\varepsilon^3) \quad \text{as} \quad \varepsilon \to 0,$$

and

$$x^4 = x_0^4 + 4\varepsilon x_1 x_0^3 + \varepsilon^2(6x_1^2 x_0^2 + 4x_0^3 x_2) + O(\varepsilon^3) \quad \text{as} \quad \varepsilon \to 0.$$

Here we have used the aforementioned rule for multiplication of asymptotic expansions.

Substituting asymptotic expressions for x, x^2 and x^4 into (4.1) using the fact that their sum is the an asymptotic expansion itself we obtain:

$$\varepsilon^2 x_0^4 - \varepsilon(x_0^2 + 2\varepsilon x_1 x_0) - x_0 - \varepsilon x_1 - \varepsilon^2 x_2 - 1 + O(\varepsilon^3) = 0 \quad \text{as} \quad \varepsilon \to 0.$$

Comparing terms of the same order of ε (ε^0, ε^1 and ε^2 respectively) we have:

$$x_0 = -1, \quad x_1 = -x_0^2 = -1, \quad x_2 = x_0^4 - 2x_0 x_1 = -1.$$

As a result of this simplest asymptotic analysis we have proven the following estimation for one of the roots:

$$x^{(1)} = -1 - \varepsilon + O(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0. \tag{4.3}$$

From the analysis one can guess the following more accurate asymptotic expression:

$$x^{(1)} = -1 - \varepsilon - \varepsilon^2 + O(\varepsilon^3), \quad \text{as} \quad \varepsilon \to 0, \tag{4.4}$$

however, to prove that $O(\varepsilon^3)$ accuracy term, one needs to improve the original assumption (4.2) up to the term $O(\varepsilon^4)$ and to repeat the same procedure again.

Thus, we have found asymptotic approximation of one root. How to find other three? It is clear that the *asymptotic ansatz* (4.2). Namely, let we assume now that:

$$x = \varepsilon^\alpha z(\varepsilon), \quad \text{as} \quad \varepsilon \to 0, \tag{4.5}$$

for some unknown up to now $\alpha \in \mathbb{R}$ and $z(0) \neq 0$ and substitute it into (4.1) to obtain:

$$\varepsilon^{2+4\alpha} z^4(\varepsilon) - \varepsilon^{1+2\alpha} z^2(\varepsilon) - \varepsilon^\alpha z(\varepsilon) - 1 = 0. \tag{4.6}$$

Let we introduce the following notations for every particular term in the equation:

$$y_1(\varepsilon, \alpha) = \varepsilon^{2+4\alpha} z^4(\varepsilon), \quad y_2(\varepsilon, \alpha) = -\varepsilon^{1+2\alpha} z^2(\varepsilon),$$

$$y_3(\varepsilon, \alpha) = -\varepsilon^\alpha z(\varepsilon), \quad y_4(\varepsilon, \alpha) = -1.$$

Then, Eq. (4.6) takes form

$$\sum_{k=1}^{4} y_k(\varepsilon, \alpha) = 0. \tag{4.7}$$

One can easy conclude that as $\varepsilon \to 0$, the following asymptotic formulas hald true:

$$y_1 = O(\varepsilon^{2+4\alpha}), \quad y_2 = O(\varepsilon^{1+2\alpha}), \quad y_3 = O(\varepsilon^{\alpha}), \quad y_4 = O(\varepsilon^0). \tag{4.8}$$

Let we first choose such value of α that the first two terms are of the same order of ε. Then $2 + 4\alpha = 1 + 2\alpha$ and thus $\alpha = -1/2$. Estimates (4.8) give

$$y_1 = O(\varepsilon^0), \quad y_2 = O(\varepsilon^0), \quad y_3 = O(\varepsilon^{-1/2}), \quad y_4 = O(\varepsilon^0), \quad \text{as} \quad \varepsilon \to 0,$$

and so the terms in Eq. (4.6) cannot be balanced.

Main idea. Let we choose α is such a way that two of these terms are of the same order as $\varepsilon \to 0$ while the other two are of a higher order.

Remark 8. Note that asymptotic representation (4.2) corresponds to the choose $\alpha = 0$ when last two terms are of zero-order with respect to the small parameter while the first two terms are of higher orders.

Let we compare the first and the third terms, then $2 + 4\alpha = \alpha$ and $\alpha = -2/3$. Substituting this new value into Eq. (4.6) we have:

$$\varepsilon^{-\frac{2}{3}} z^4 - \varepsilon^{-\frac{1}{3}} z^2 - \varepsilon^{-\frac{2}{3}} z - 1 = 0,$$

or

$$z^4 - z - \varepsilon^{\frac{1}{3}} z^2 - \varepsilon^{\frac{2}{3}} = 0.$$

Let we now introduce a new small parameter:

$$\nu = \varepsilon^{\frac{1}{3}}, \tag{4.9}$$

then the algebraic equation (4.1) takes form

$$z^4 - \nu z^2 - z - \nu^2 = 0. \tag{4.10}$$

Solution to this equation again will be sought with use of the straightforward asymptotic procedure. Let us assume that

$$z = z_0 + \nu z_1 + \nu^2 z_2 + O(\nu^3) \quad \text{as} \quad \nu \to 0, \tag{4.11}$$

with $z_0 \neq 0$. Again, we can compute

$$z^2 = z_0^2 + 2\nu z_1 z_0 + \nu^2 (z_1^2 + 2 z_0 z_2) + O(\nu^3) \quad \text{as} \quad \nu \to 0,$$

$$z^4 = z_0^4 + 4\nu z_1 z_0^3 + \nu^2(6z_1^2 z_0^2 + 4z_0^3 z_2) + O(\nu^3) \quad \text{as} \quad \nu \to 0.$$

Substituting this into Eq. (4.10) and balancing terms of the same order with respect to the new small parameter ν we have the following three identities:

$$z_0^4 - z_0 = 0, \quad 4z_1 z_0^3 - z_0^2 - z_1 = 0, \quad 6z_1^2 z_0^2 + 4z_0^3 z_2 - 2z_1 z_0 - z_2 - 1 = 0.$$

Since z_0 cannot be zero, one concludes

$$z_0^3 = 1, \tag{4.12}$$

and the next two equations can be simplified allowing to compute two remaining constants z_1 and z_2:

$$z_1 = \frac{1}{3} z_0^2 = \frac{1}{3z_0}, \quad z_2 = \frac{1}{3}\left(1 + 2z_0 z_1 - 6z_0^2 z_1^2\right) = \frac{1}{3}.$$

The first solution to Eq. (4.12) is immediate $z_0 = 1$ and the respective asymptotic expansion (4.11) takes form:

$$z^{(1)}(\nu) = 1 + \frac{1}{3}\nu + \frac{1}{3}\nu^2 + \ldots, \quad \text{as} \quad \nu \to 0. \tag{4.13}$$

Other two solutions of Eq. (4.12) are complex conjugate:

$$z_0^{\pm} = e^{\pm\frac{2\pi i}{3}} = \cos\frac{2\pi}{3} \pm i \sin\frac{2\pi}{3} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Note also that $z_0^+ = \bar{z}_0^-$ and $(z_0^+)^2 = z_0^-$. This allows us to construct asymptotic expansions (4.11) for last two solutions:

$$z^{\pm}(\nu) = 1 + \frac{1}{3} z_0^{\mp}\nu + \frac{1}{3}\nu^2 + \ldots, \quad \text{as} \quad \nu \to 0. \tag{4.14}$$

As a result, we have now full picture to reconstruct three zeros of the original algebraic equation (4.1)

$$x^{(2)} = \varepsilon^{-2/3} + \frac{1}{3}\varepsilon^{-1/3} + O(1), \quad \text{as} \quad \varepsilon \to 0, \tag{4.15}$$

$$x^{(3)} = \varepsilon^{-2/3} + \frac{1}{3} z_0^+ \varepsilon^{-1/3} + O(1), \quad x^{(4)} = \varepsilon^{-2/3} + \frac{1}{3} z_0^- \varepsilon^{-1/3} + O(1), \quad \text{as} \quad \varepsilon \to 0. \tag{4.16}$$

In case if for $\alpha = -2/3$ less than three solutions were found we would need to search for another appropriate value of α.

One can ask a question: is there a more rational method to find the terms which balance each other? Or in other words, how to find the dominant balance allowing to define values of the parameter α which turns the equation into a form more suitable for asymptotic analysis.

1.4.2 Dominant balance

Let us write an algebraic equation in a general form:

$$\sum_{p,q} C_{p,q}\varepsilon^q x^p = 0. \tag{4.1}$$

Thus, for Eq. (4.1) we have $p = 0, 1, 2, 3, 4$, $q = 0, 1, 2$

$$C_{4,2} = 1, \quad C_{2,1} = -1, \quad C_{1,0} = -1, \quad C_{0,0} = -1,$$

while for all other combinations of the indices $C_{p,q} = 0$. As previously we search for asymptotic solution of the equation (4.1) in the form (4.5) and substituting it in (4.1) we obtain:

$$\sum_{p,q} C_{p,q}\varepsilon^{q+\alpha p} z^p(\varepsilon) = 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.2}$$

Since $z(0) \neq 0$, terms of the same order in ε in (4.2) are represented by a line

$$q + \alpha p = \beta, \tag{4.3}$$

where β is an unknown constant. Each coefficient $C_{p,q}$ of the polynomial (4.1) can be depicted as a point in the (p, q) plane (see Fig. 4.10). Below this line (so $q + \alpha p < \beta$ the power is smaller than on the line. Therefore the corresponding term is of lower order than for the terms (if any) corresponding to this line. Above this line, the corresponding terms have higher orders than that on the line.

Conclusion. To find all possible combinations of dominant balances for a given polynomial, find all possible placements of a line so that it includes two or more terms of the polynomials, with all other points lying above this line. Graphically, this corresponds to bringing the line up from below until it makes contact with a point, and then rotating it one way or the other until it makes contact with a second point. Such a plot is known as a Kruskal-Newton graph.

In Fig. 4.10 we present by red lines two options leading to the solutions ($\alpha = 0$ and $\alpha = -2/3$) while by green line we indicated the wrong way ($\alpha = -1/3$).

Remark 9. Note that balance dominant approach is useful not only for algebraic equation but to any type of equations which can be written in

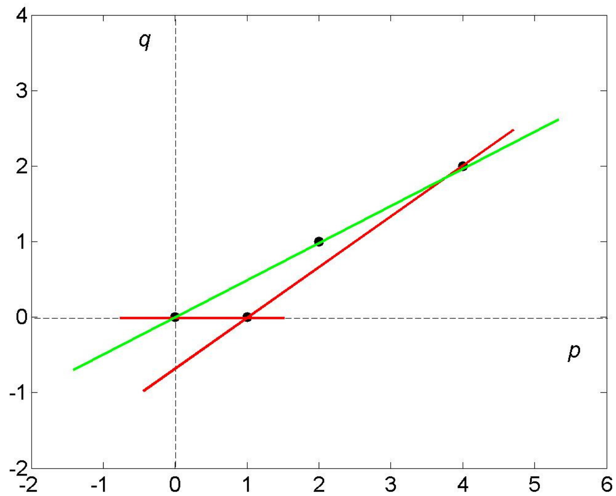

Figure 4.10: Kruskal—Newton graph for Eq. (4.1).

form (4.1) with an arbitrary operators (for example differential and integral operators) $C_{p,q}$ acting on unknown function x which should be find in the asymptotic procedure. However, as it will be seen later if there is no many terms in Eq. (4.1), the proper rescaling can be found even without use of the Kruskal-Newton graph.

Chapter 2

Asymptotic methods in nonlinear oscillations

2.1 Lindstedt — Poincaré method

2.1.1 Free undamped oscillations

Let us consider oscillations of a single mass point attached to a nonlinear spring. According to Newton's second law, the equation of motion of the particle is

$$m\ddot{x} + F(x) = 0. \tag{1.1}$$

Here, m is the mass of the particle, x is the coordinate of the mass point, $-F(x)$ is the value of the force exerted by the string.

For a nonlinear spring, one can introduce the *stiffness* of the spring (for the displacement x) as the derivative $F'(x)$ of function $F(x)$ with respect to x. If the stiffness increases with the displacement (see. Fig. 1.2 a), then the spring (or the corresponding elastic force) is called *stiff*. If, in turn, the stiffness decreases with increasing displacement (see. Fig. 1.2 b), then the elastic force (of the nonlinear spring) is called *complaint*.

Since Eq. (1.1) does not explicitly depend on the time variable t, using the substitution

$$v = \frac{dx}{dt}, \quad \frac{d^2x}{dt^2} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

we reduce Eq. (1.1) to the following separable differential equation:

$$mv\,dv = -F(x)\,dx.$$

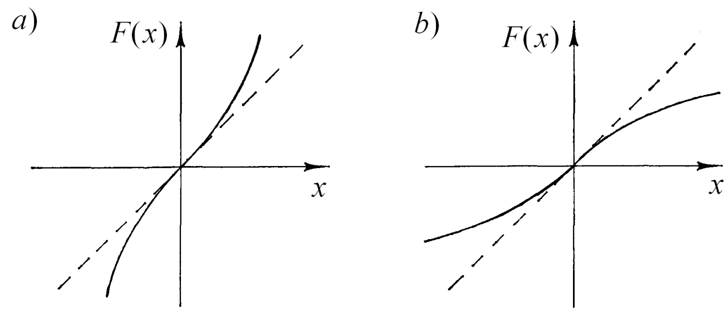

Figure 1.1: Stiff and complaint elastic forces.

Now assuming that $v = v_0$ for $x = x_0$ at the time moment $t = t_0$, we find

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = -\int_{x_0}^{x} F(\xi)\, d\xi. \tag{1.2}$$

Eq. (1.2) reflects the law of conservation of mechanical energy. The left-hand side of Eq. (1.2) represents the variation of the kinetic energy of the oscillating particle, while its right-hand side is the mechanical work of the restoring force, i. e., the variation of the potential energy.

2.1.2 Duffing equation. Straightforward asymptotic expansion method

Consider small oscillations of a single mass point of mass m in the case of symmetrical nonlinear restoring force

$$F(x) = kx + k_3 x^3.$$

Substituting this expression into Eq. (1.1), we rewrite it in the form

$$\frac{d^2x}{dt^2} + \omega_0^2 x + \frac{k_3}{m} x^3 = 0; \quad \omega_0 = \sqrt{\frac{k}{m}}. \tag{1.3}$$

Here, ω_0 is the angular frequency of linear oscillations in the case $k_3 = 0$.

We are looking for solutions of Eq. (1.3) satisfying the initial conditions

$$x(0) = a, \quad \frac{dx}{dt}(0) = 0. \tag{1.4}$$

Assuming that the deviation x of the mass point from the equilibrium position $x = 0$ is relatively small, we put

$$a = \varepsilon A, \tag{1.5}$$

where ε is a small dimensionless parameter, A is a fixed value which does not vary with ε.

By making the change of variable

$$x = \varepsilon \xi, \tag{1.6}$$

we obtain that according to (1.4) and (1.5), the function $\xi(t)$ will satisfy the initial conditions

$$\xi(0) = A, \quad \frac{d\xi}{dt}(0) = 0. \tag{1.7}$$

Substituting the expression (1.6) into the nonlinear differential equation (1.3), we get

$$\frac{d^2\xi}{dt^2} + \omega_0^2 \xi + \mu \xi^3 = 0, \tag{1.8}$$

where $\mu = \varepsilon^2 k_3/m$ is a new small parameter.

Let us try to construct the solution of the problem (1.8), (1.7) by a regular perturbation expansion with respect to parameter μ as follows:

$$\xi(t) = \xi_0(t) + \mu \xi_1(t) + \mu^2 \xi_2(t) + \dots\,. \tag{1.9}$$

Differentiating twice with respect to t, we get

$$\frac{d^2\xi}{dt^2} = \frac{d^2\xi_0}{dt^2} + \mu \frac{d^2\xi_1}{dt^2} + \mu^2 \frac{d^2\xi_2}{dt^2} + \dots\,. \tag{1.10}$$

By multiplying the power series, one gets

$$\begin{aligned} \xi^2 &= \bigl(\xi_0 + \mu\xi_1 + \mu^2\xi_2 + \dots\bigr)\bigl(\xi_0 + \mu\xi_1 + \mu^2\xi_2 + \dots\bigr) \\ &= \xi_0^2 + 2\mu\xi_0\xi_1 + \mu^2(2\xi_0\xi_2 + \xi_1^2) + \dots; \end{aligned} \tag{1.11}$$

$$\begin{aligned} \xi^3 &= \bigl[\xi_0^2 + 2\mu\xi_0\xi_1 + \mu^2(2\xi_0\xi_2 + \xi_1^2) + \dots\bigr]\bigl(\xi_0 + \mu\xi_1 + \mu^2\xi_2 + \dots\bigr) \\ &= \xi_0^3 + 3\mu\xi_0^2\xi_1 + 3\mu^2(\xi_0^2\xi_2 + \xi_0\xi_1^2) + \dots\,. \end{aligned} \tag{1.12}$$

Now, substituting the expansions (1.9), (1.10), and (1.12) into Eq. (1.8), we obtain the power series with respect to μ, which should be identically equal to zero. Thus, we arrive at a system of equations, the first two of which are as follows:

$$\ddot{\xi}_0 + \omega_0^2 \xi_0 + \mu(\ddot{\xi}_1 + \omega_0^2 \xi_1 + \xi_0^3) + \ldots = 0. \tag{1.13}$$

Hence, all the coefficients at the successive powers of μ should be equal to zero, i. e.,

$$\ddot{\xi}_0 + \omega_0^2 \xi_0 = 0, \tag{1.14}$$
$$\ddot{\xi}_1 + \omega_0^2 \xi_1 = -\xi_0^3. \tag{1.15}$$

Correspondingly, the initial conditions (1.7) lead to the following equations:

$$\xi_0(0) = A, \quad \dot{\xi}_0(0) = 0; \tag{1.16}$$

$$\xi_i(0) = 0, \quad \dot{\xi}_i(0) = 0 \quad (i = 1, 2, \ldots). \tag{1.17}$$

Thus, the original problem (1.8), (1.7) has been reduced to a sequence of linear problems (1.14), (1.16); (1.15), (1.17), $i = 1; \ldots$.

It is readily seen that Eq. (1.14) with the initial conditions (1.16) taken into account has the following solution:

$$\xi_0(t) = A \cos \omega_0 t. \tag{1.18}$$

Consequently, Eq. (1.15) takes the form

$$\ddot{\xi}_1 + \omega_0^2 \xi_1 = -\frac{3}{4} A^3 \cos \omega_0 t - \frac{1}{4} A^3 \cos 3\omega_0 t. \tag{1.19}$$

Here we used the formula $\cos^3 \varphi = (1/4) \cos 3\varphi + (3/4) \cos \varphi$.

Eq. (1.19) with the initial conditions (1.17) taken into account has the following solution:

$$\xi_1(t) = -\frac{3}{8\omega_0} A^3 t \sin \omega_0 t + \frac{A^3}{32\omega_0^2} \big[\cos 3\omega_0 t - \cos \omega_0 t\big]. \tag{1.20}$$

Note that the terms like $t \sin \omega_0 t$ which increase unboundedly as $t \to \infty$ are called *secular*.

Thus, according to the solutions (1.18) and (1.20), we obtain the following first-order approximation:

$$\xi(t) \approx A\cos\omega_0 t + \mu\left(-\frac{3}{8\omega_0}A^3 t\sin\omega_0 t + \frac{A^3}{32\omega_0^2}\left[\cos 3\omega_0 t - \cos\omega_0 t\right]\right). \quad (1.21)$$

Formula (1.21) has an error of $O(\mu^2 t^2)$. Indeed, the first-order correction term $\mu\xi_1(t)$ will be small with respect to $\xi_0(t)$, as it was assumed in (1.9), then, and only then, when the quantity μt is small compared to unit (see formula (1.21)). Hence, the straightforward expansion (1.9) turns out to be valid only on a finite time interval $0 \leq t \leq t_1$ with some t_1, where $\mu t \ll 1$.

We underline that the solution (1.21) obtained by a straightforward expansion method is not uniformly suitable for the whole range of independent variable t. Such asymptotic expansions are called *inhomogeneous with respect to t*.

2.1.3 Duffing equation. Method of "elongated" parameters

Once again, let us consider the Duffing differential equation

$$\frac{d^2x}{dt^2} + \omega_0^2 x + \mu x^3 = 0, \quad (1.22)$$

where $0 < \mu$ is a small parameter.

Since we are looking for periodic solutions, the time reference point can be chosen arbitrarily. So, let us assume that at the initial moment of time t the following conditions take place:

$$x(0) = A, \quad \frac{dx}{dt}(0) = 0. \quad (1.23)$$

For $\mu = 0$, the general solution of Eq. (1.22) is a linear combination of the functions $\sin\omega_0 t$ and $\cos\omega_0 t$ with the period $2\pi/\omega_0$. For $\mu \neq 0$, the period of the solution to Eq. (1.22) becomes unknown. That is why, it reasonable to change the time variable from t to a new independent variable

$$\tau = \omega t, \quad (1.24)$$

where ω is an unknown angular frequency of the sought-for periodic solution. Note that ω is a constant, which is determined only by the values of the parameter μ and amplitude A). It is clear that the function x will now have the period 2π with respect to variable τ.

By the chain rule of differentiation, we get

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \omega\frac{d}{d\tau}, \quad \frac{d^2}{dt^2} = \omega^2\frac{d^2}{d\tau^2}.$$

Then, after this change of independent variable, Eq. (1.22) takes the form

$$\omega^2\frac{d^2x}{d\tau^2} + \omega_0^2 x + \mu x^3 = 0. \tag{1.25}$$

Method of "elongated" parameters (or the Lindstedt — Poincaré method) consists in the following. The sought-for periodic solution $x(\tau)$ is broken down into a power series in μ with the coefficients being periodic functions of τ with the period 2π. Thus, we put

$$x(\tau) = x_0(\tau) + \mu x_1(\tau) + \mu^2 x_2(\tau) + \dots, \tag{1.26}$$

where $x_i(\tau)$ is a 2π-periodic function of τ, i. e.,

$$x_i(\tau + 2\pi) = x_i(\tau). \tag{1.27}$$

Besides the function x, it is also necessary to expand the quantity ω in ascending powers of the parameter μ, that is

$$\omega = \omega_0 + \mu\omega_1 + \mu^2\omega_2 + \dots. \tag{1.28}$$

Substituting the expansions (1.26), (1.28), (1.12), and

$$\omega^2 = \omega_0^2 + 2\mu\omega_0\omega_1 + \mu^2(2\omega_0\omega_2 + \omega_1^2) + \dots \tag{1.29}$$

into Eq. (1.25), we obtain a power series with respect to μ, which should be identically equal to zero. Equating the coefficients of successive powers of μ to zero gives a series of linear differential equations

$$\begin{aligned} \frac{d^2x_0}{d\tau^2} + x_0 &= 0, \\ \frac{d^2x_1}{d\tau^2} + x_1 &= -\frac{2\omega_1}{\omega_0}\frac{d^2x_0}{d\tau^2} - \frac{1}{\omega_0^2}x_0^3, \\ \frac{d^2x_2}{d\tau^2} + x_2 &= -\left(\frac{2\omega_2}{\omega_0} + \frac{\omega_1^2}{\omega_0^2}\right)\frac{d^2x_0}{d\tau^2} - \frac{2\omega_1}{\omega_0}\frac{d^2x_1}{d\tau^2} - \frac{3}{\omega_0^2}x_0^2x_1, \\ &\dots \end{aligned} \tag{1.30}$$

According to the initial conditions (1.23), we derive the following series of initial conditions:

$$x_i(0) = \delta_{i0} A, \quad \frac{dx_i}{d\tau}(0) = 0 \quad (i = 0, 1, \ldots). \tag{1.31}$$

Here, δ_{ij} is Kronecker's delta, i. e., $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

Let us proceed to sequential solving the problems $(1.30)_i$, $(1.31)_i$. It is evident that the solution $(1.30)_1$ subject to the initial conditions $(1.31)_1$ is

$$x_0(\tau) = A \cos \tau. \tag{1.32}$$

Thus, Eq. $(1.30)_2$ takes the form

$$\frac{d^2 x_1}{d\tau^2} + x_1 = \left(\frac{2\omega_1}{\omega_0} - \frac{3A^2}{4\omega_0^2} \right) A \cos \tau - \frac{A^3}{4\omega_0^2} \cos 3\tau. \tag{1.33}$$

If the coefficient at $\cos \tau$ on the right-hand side of Eq. (1.33) is not equal to zero, then the solution to Eq. (1.33) will contain the expression $\tau \cos \tau$, i. e., a secular term. Therefore, the periodic condition (1.27) for the function $x_1(\tau)$ requires that this coefficient be equal to zero, i. e.,

$$\omega_1 = \frac{3}{8} \frac{A^2}{\omega_0}. \tag{1.34}$$

Consequently, taking into account the initial conditions $(1.31)_2$, we obtain the solution to Eq. (1.33) in the form

$$x_1(\tau) = \frac{1}{32} \frac{A^3}{\omega_0^2} \big(- \cos \tau + \cos 3\tau \big). \tag{1.35}$$

Thus, based on formulas (1.24), (1.26), (1.32), and (1.35), we obtain the solution to Eq. (1.22) with the accuracy up to the terms of the second order with respect to μ in the form of the expansion

$$x(\tau) = A \cos \omega t + \mu \frac{1}{32} \frac{A^3}{\omega_0^2} \big(- \cos \tau + \cos 3\tau \big) + \ldots\,, \tag{1.36}$$

where the angular frequency is determined from the relations (1.28) and (1.34) as follows:

$$\omega = \omega_0 + \mu \frac{3}{8} \frac{A^2}{\omega_0} + \ldots\,. \tag{1.37}$$

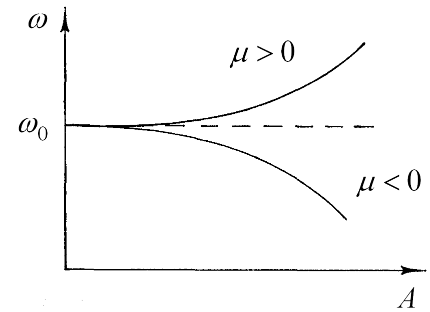

Figure 1.2: Amplitude/frequency curves for stiff and complaint elastic forces.

Consequently, the period of oscillations is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0}\left(1 - \mu\frac{3}{8}\frac{A^2}{\omega_0^2} + \dots\right).$$

It is clear that the period of oscillations depends on the amplitude of oscillations. For a stiff elastic force ($\mu > 0$), the period of oscillations decreases (the oscillation frequency increases) with increasing the amplitude; and, on the contrary, for a complaint elastic force ($\mu < 0$) the period of oscillations increases (the oscillation frequency decreases) with increasing the amplitude. Fig. 2 schematically represents the amplitude/frequency curves showing the relationship between the amplitude A and the angular frequency ω in the cases of stiff and complaint elastic forces. These curves have a joint tangent at $\omega = \omega_0$, which is the frequency at $\mu = 0$ corresponding to a linear restoring elastic force.

The process of constructing of approximate solution to Eq. (1.22) can be continued. In particular, the third term of the expansion (1.26) can be written as follows:

$$\omega_2 = -\frac{21}{256}\frac{A^4}{\omega_0^3},$$

$$x_2(\tau) = \frac{A^5}{\omega_0^4}\left(\frac{23}{1024}\cos\tau - \frac{3}{128}\cos 3\tau + \frac{1}{1024}\cos 5\tau\right).$$

Hence, the solution to Eq. (1.22) with the accuracy up to the terms of the third order with respect to μ takes the following form (see also [14], § 1.2):

$$x(\tau) = \left(A - \mu\frac{1}{32}\frac{A^3}{\omega_0^2} + \mu^2\frac{23}{1024}\frac{A^5}{\omega_0^4}\right)\cos\omega t +$$

$$+\left(\mu\frac{1}{32}\frac{A^3}{\omega_0^2}-\mu^2\frac{3}{128}\frac{A^5}{\omega_0^4}\right)\cos 3\omega t+\mu^2\frac{1}{1024}\frac{A^5}{\omega_0^4}\cos 5\omega\tau+\dots . \quad (1.38)$$

Here the angular frequency is given by

$$\omega=\omega_0+\mu\frac{3}{8}\frac{A^2}{\omega_0}-\mu^2\frac{21}{256}\frac{A^4}{\omega_0^3}\dots . \quad (1.39)$$

In turn, the obtained solution (1.38), (1.39) can be improved through computations of successive approximations.

2.1.4 Lindstedt — Poincaré method. Quasilinear undamped oscillations

Consider now the nonlinear differential equation

$$\frac{d^2x}{dt^2}+\omega_0^2x+\mu f(x)=0 \quad (1.40)$$

with the initial conditions

$$x(0)=A, \quad \frac{dx}{dt}(0)=0. \quad (1.41)$$

It is clear that the Duffing equation studied in the previous sections is a special case of Eq. (1.40).

Assuming that the quantity μ is small, let us construct an approximate solution to the problem (1.40), (1.41) using the Lindstedt — Poincaré method.

First, by means of the change of independent variable

$$\tau=\omega t,$$

where ω is an unknown angular frequency of the sought-for solution, Eq. (1.40) can be transformed as follows:

$$\omega^2\frac{d^2x}{d\tau^2}+\omega_0^2x+\mu f(x)=0. \quad (1.42)$$

Second, we represent the solution to Eq. (1.42) in the form of expansion

$$x(\tau,\mu)=x_0(\tau)+\mu x_1(\tau)+\mu^2x_2(\tau)+\dots , \quad (1.43)$$

where $x_i(\tau)$ are 2π-periodic functions of the variable τ. Moreover, the quantity ω has to be also expanded into a series as

$$\omega = \omega_0 + \mu\omega_1 + \mu^2\omega_2 + \dots . \tag{1.44}$$

Further, we substitute the expansions (1.43) and (1.44) into Eq. (1.42). To do this we apply the formula

$$f\big(x(\tau,\mu)\big) = f\big(x(\tau,0)\big) + \frac{\partial f}{\partial \mu}\big(x(\tau,\mu)\big)\bigg|_{\mu=0} \mu + \frac{1}{2!}\,\frac{\partial^2 f}{\partial \mu^2}\big(x(\tau,\mu)\big)\bigg|_{\mu=0} \mu^2 + \dots .$$

By the chain rule of differentiation, we get

$$\frac{\partial f}{\partial \mu}\big(x(\tau,\mu)\big) = f'_x\big(x(\tau,\mu)\big)\frac{\partial x}{\partial \mu}(\tau,\mu),$$

$$\frac{\partial^2 f}{\partial \mu^2}\big(x(\tau,\mu)\big) = f''_{xx}\big(x(\tau,\mu)\big)\left[\frac{\partial x}{\partial \mu}(\tau,\mu)\right]^2 + f'_x\big(x(\tau,\mu)\big)\frac{\partial^2 x}{\partial \mu^2}(\tau,\mu).$$

Now, differentiating (formally) the expansion (1.43), we find

$$\frac{\partial x}{\partial \mu}(\tau,\mu) = x_1(\tau) + 2\mu x_2(\tau) + \dots , \quad \frac{\partial^2 x}{\partial \mu^2}(\tau,\mu) = 2x_2(\tau) + \dots .$$

Hence, under the assumption of sufficient differentiability of the function $f(x)$, we obtain

$$f\big(x(\tau,\mu)\big) = f(x_0) + \mu f'_x(x_0)x_1 + \mu^2\Big(\frac{1}{2}f''_{xx}(x_0)x_1^2 + f'_x(x_0)x_2\Big) + \dots . \tag{1.45}$$

Thus, the substitution of the expansions (1.43) and (1.44) into Eq. (1.42) with the expansion (1.45) taken into account yields a power series with respect to the parameter μ, which should be identically equal to zero. Equating the coefficients of the successive powers of μ to zero, we derive the following system of differential equations:

$$\begin{aligned}
\frac{d^2x_0}{d\tau^2} + x_0 &= 0,\\
\frac{d^2x_1}{d\tau^2} + x_1 &= -\frac{1}{\omega_0^2}f(x_0) - \frac{2\omega_1}{\omega_0}\frac{d^2x_0}{d\tau^2},\\
\frac{d^2x_2}{d\tau^2} + x_2 &= -\frac{1}{\omega_0^2}f'_x(x_0)x_1 - \left(\frac{2\omega_2}{\omega_0} + \frac{\omega_1^2}{\omega_0^2}\right)\frac{d^2x_0}{d\tau^2} - \frac{2\omega_1}{\omega_0}\frac{d^2x_1}{d\tau^2},\\
&\dots
\end{aligned} \tag{1.46}$$

Correspondingly, the initial conditions (1.41) lead to the following initial conditions:

$$x_i(0) = \delta_{i0} A, \quad \frac{dx_i}{d\tau}(0) = 0 \quad (i = 0, 1, \ldots). \tag{1.47}$$

Here, δ_{i0} is the Kronecker symbol.

The solution of the limit ($\mu = 0$) problem $(1.46)_1$, $(1.47)_1$ is evident:

$$x_0(\tau) = A \cos \tau.$$

Substituting this expression into Eq. $(1.46)_2$, we obtain

$$\frac{d^2 x_1}{d\tau^2} + x_1 = -\frac{1}{\omega_0^2} f(A\cos\tau) + \frac{2\omega_1}{\omega_0} A \cos\tau. \tag{1.48}$$

Let us now determine the conditions under which Eq. (1.48) admits a periodic solution. Since the function $f(A\cos\tau)$ is even and periodic with the period 2π, it can be expanded into the Fourier series containing only $\cos k\tau$, i. e.,

$$f(A\cos\tau) = \frac{a_0(A)}{2} + \sum_{k=1}^{\infty} a_k(A) \cos k\tau,$$

where the Fourier coefficients are calculated by the formula

$$a_k(A) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(A\cos\tau) \cos k\tau \, d\tau.$$

Hence, in the right-hand side of Eq. (1.48), there will be the term

$$\left(-\frac{1}{\omega_0^2} a_1(A) + \frac{2\omega_1}{\omega_0} A \right) \cos\tau, \tag{1.49}$$

generating a secular term in the particular solution. Equating to zero the coefficient of $\cos\tau$ in (1.49), we obtain the equation

$$-\frac{1}{\pi\omega_0^2} \int_{-\pi}^{\pi} f(A\cos\tau) \cos\tau \, d\tau + \frac{2\omega_1}{\omega_0} A = 0,$$

from which one can determine the first frequency correction

$$\omega_1 = \frac{1}{2\pi A \omega_0} \int_{-\pi}^{\pi} f(A\cos\tau) \cos\tau \, d\tau.$$

The solution to Eq. (1.48) satisfying the initial conditions $(1.47)_2$ can be represented in the integral form

$$x_1(\tau) = \int\limits_0^\tau \left[-\frac{1}{\omega_0^2} f(A\cos\sigma) + \frac{2\omega_1}{\omega_0} A\cos\sigma \right] \sin(\tau - \sigma)\, d\sigma.$$

Finally, observe that the Lindstedt — Poincaré method allows us to study only steady regimes, because using this method, one can construct only periodic solutions. In particular, the Lindstedt — Poincaré method is not appropriate for considering transient processes in damped oscillations systems.

2.2 Two-scale asymptotic method

2.2.1 Free damped vibrations

Consider the motion of a single mass point of mass m attached to a nonlinear spring in a damping medium according the following differential equation:

$$m\ddot{x} + \varphi(\dot{x}) + F(x) = 0. \tag{2.1}$$

Here, $-F(x)$ is an elastic restoring force exerted by the spring, $-\varphi(\dot{x})$ is a damping force.

Recall that a force $-\varphi(\dot{x})$ depending only on the velocity $\dot{x}$ is called *dissipative,* if for any velocity $\dot{x} \neq 0$ the following condition is satisfied:

$$-\varphi(\dot{x})\dot{x} < 0. \tag{2.2}$$

Let us multiply the both sides of Eq. (2.1) by $\dot{x}$. The obtained relation can be written in the form

$$\frac{dE}{dt}(x, \dot{x}) = -\varphi(\dot{x})\dot{x}, \tag{2.3}$$

where $E(x, \dot{x})$ is the total energy of the oscillating point, i. e.,

$$E(x, \dot{x}) = \frac{m\dot{x}^2}{2} + \int\limits_{x_0}^{x} F(\xi)\, d\xi.$$

It is clear now that since the right-hand side of Eq. (2.3) satisfies the inequality (2.2), the total mechanical energy of the mass point moving in the damping medium energy may only decrease, that is

$$\dot{E}(x, \dot{x}) \leq 0. \tag{2.4}$$

Moreover, the equality sign in (2.4) takes place only at such time moments when $\dot{x} = 0$. Thus, the oscillations performing under the action of a conservative and a dissipative forces decay with time.

2.2.2 Linear oscillator with small damping

Consider the problem of free oscillations with damping proportional to the velocity

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \tag{2.5}$$

$$x(0) = A, \quad \dot{x}(0) = 0. \tag{2.6}$$

Eq. (2.5) describes the mechanical motion which is characterized by two time scales: the period of free undamped oscillations $2\pi/\omega_0$ and the relaxation time $1/\beta$. The main idea of the two-scale method consists in explicit distinguishing the time variables related to these two time scales.

So, introducing dimensionless variables, we put

$$\tau = \omega_0 t, \quad \mathcal{T} = \beta t. \tag{2.7}$$

Assuming that the dimensionless parameter

$$\varepsilon = \frac{\beta}{\omega_0} \tag{2.8}$$

is small, we will construct the solution to the problem (2.5), (2.6) in the following form:

$$x = x_0(\tau, \mathcal{T}) + \varepsilon x_1(\tau, \mathcal{T}) + \varepsilon^2 x_2(\tau, \mathcal{T}) + \dots . \tag{2.9}$$

We underline that the variables τ and $\mathcal{T}$ are assumed to be independent.

By the chain rule of differentiation, we will have

$$\frac{dx_k}{dt} = \omega_0 \frac{\partial x_k}{\partial \tau} + \beta \frac{\partial x_k}{\partial \mathcal{T}}, \tag{2.10}$$

$$\frac{d^2 x_k}{dt^2} = \omega_0^2 \frac{\partial^2 x_k}{\partial \tau^2} + 2\beta\omega_0 \frac{\partial^2 x_k}{\partial \tau \partial \mathcal{T}} + \beta^2 \frac{\partial^2 x_k}{\partial \mathcal{T}^2}. \tag{2.11}$$

Differentiating the expansion (2.9) term-by-term and collecting similar terms, we obtain

$$\frac{dx}{dt} = \omega_0 \left(\sum_{k=0}^{\infty} \varepsilon^k \frac{\partial x_k}{\partial \tau} + \sum_{k=0}^{\infty} \varepsilon^{k+1} \frac{\partial x_k}{\partial \mathcal{T}} \right).$$

Here we used the relation $\beta = \varepsilon\omega_0$.

In the last sum above, we make use of the change of the running index $k = l - 1$ as follows:

$$\frac{dx}{dt} = \omega_0 \left(\sum_{k=0}^{\infty} \varepsilon^k \frac{\partial x_k}{\partial \tau} + \sum_{l=1}^{\infty} \varepsilon^{l} \frac{\partial x_{l-1}}{\partial \mathcal{T}} \right).$$

Now, denoting the umbral index l by the symbol k, we finally obtain

$$\frac{dx}{dt} = \omega_0 \frac{\partial x_0}{\partial \tau} + \omega_0 \sum_{k=1}^{\infty} \varepsilon^k \left(\frac{\partial x_k}{\partial \tau} + \frac{\partial x_{k-1}}{\partial \mathcal{T}} \right). \tag{2.12}$$

In an analogous way, in addition to (2.12), we derive the following expansion:

$$\begin{aligned} \frac{1}{\omega_0^2}\frac{d^2x}{dt^2} &= \frac{\partial^2 x_0}{\partial \tau^2} + \varepsilon \left(\frac{\partial^2 x_1}{\partial \tau^2} + 2\frac{\partial^2 x_0}{\partial \tau \partial \mathcal{T}} \right) \\ &\quad + \sum_{k=2}^{\infty} \varepsilon^k \left(\frac{\partial^2 x_k}{\partial \tau^2} + 2\frac{\partial^2 x_{k-1}}{\partial \tau \partial \mathcal{T}} + \frac{\partial^2 x_{k-2}}{\partial \mathcal{T}^2} \right). \end{aligned} \tag{2.13}$$

Substituting the expansions (2.9), (2.12), and (2.13) into Eq. (2.5) and the initial conditions (2.6), after some algebra we will have

$$\begin{aligned} &\frac{\partial^2 x_0}{\partial \tau^2} + x_0 + \varepsilon \left(\frac{\partial^2 x_1}{\partial \tau^2} + 2\frac{\partial^2 x_0}{\partial \tau \partial \mathcal{T}} + 2\frac{\partial x_0}{\partial \tau} + x_1 \right) \\ &+ \sum_{k=2}^{\infty} \varepsilon^k \left(\frac{\partial^2 x_k}{\partial \tau^2} + 2\frac{\partial^2 x_{k-1}}{\partial \tau \partial \mathcal{T}} + \frac{\partial^2 x_{k-2}}{\partial \mathcal{T}^2} + 2\frac{\partial x_{k-1}}{\partial \tau} + 2\frac{\partial x_{k-2}}{\partial \mathcal{T}} + x_k \right) = 0; \end{aligned} \tag{2.14}$$

$$x_0(0,0) + \sum_{k=1}^{\infty} \varepsilon^k x_k(0,0) = A, \tag{2.15}$$

$$\frac{\partial x_0}{\partial \tau}(0,0) + \sum_{k=1}^{\infty} \varepsilon^k \left(\frac{\partial x_k}{\partial \tau}(0,0) + \frac{\partial x_{k-1}}{\partial \mathcal{T}}(0,0) \right) = 0. \tag{2.16}$$

Equating to zero the coefficients of powers of the parameter ε in the expansion (2.14), we derive the following recurrence sequence of equations:

$$\frac{\partial^2 x_0}{\partial \tau^2} + x_0 = 0, \tag{2.17}$$

$$\frac{\partial^2 x_1}{\partial \tau^2} + x_1 = -2\frac{\partial^2 x_0}{\partial \tau \partial \mathcal{T}} - 2\frac{\partial x_0}{\partial \tau}, \tag{2.18}$$

$$\frac{\partial^2 x_2}{\partial \tau^2} + x_2 = -2\frac{\partial^2 x_1}{\partial \tau \partial \mathcal{T}} - \frac{\partial^2 x_0}{\partial \mathcal{T}^2} - 2\frac{\partial x_1}{\partial \tau} - 2\frac{\partial x_0}{\partial \mathcal{T}}, \tag{2.19}$$

$$\dots$$

Consequently, from the initial conditions (2.15), (2.16) it follows that

$$x_0(0,0) = A, \quad \frac{\partial x_0}{\partial \tau}(0,0) = 0, \tag{2.20}$$

$$x_k(0,0) = 0, \quad \frac{\partial x_k}{\partial \tau}(0,0) = -\frac{\partial x_{k-1}}{\partial \mathcal{T}}(0,0) \quad (k = 1, 2, \dots). \tag{2.21}$$

Since the relations (2.17)–(2.19) are linear partial differential equations with the right-hand sides depending only on the derivatives with respect to the variable τ, they can be integrated in the same way as ordinary differential equations with the only difference that the constants of integration should be regarded as functions of the variable $\mathcal{T}$. Therefore, the solution to Eq. (2.17) takes the form

$$x_0 = A_0(\mathcal{T}) \cos \tau + B_0(\mathcal{T}) \sin \tau. \tag{2.22}$$

The substitution of the function (2.22) into the relations (2.20) gives the initial conditions for the functions $A_0(\mathcal{T})$ and $B_0(\mathcal{T})$, i. e.,

$$A_0(0) = A, \quad B_0(0) = 0. \tag{2.23}$$

Further, according to the expression (2.22), Eq. (2.18) takes the form

$$\frac{\partial^2 x_1}{\partial \tau^2} + x_1 = 2 \left[\left(\frac{dA_0}{d\mathcal{T}} + A_0 \right) \sin \tau - \left(\frac{dB_0}{d\mathcal{T}} + B_0 \right) \cos \tau \right]. \tag{2.24}$$

The general solution to Eq. (2.24) is as follows:

$$x_1 = A_1(\mathcal{T})\cos\tau + B_1(\mathcal{T})\sin\tau$$
$$-\tau\left[\left(\frac{dA_0}{d\mathcal{T}} + A_0\right)\cos\tau + \left(\frac{dB_0}{d\mathcal{T}} + B_0\right)\sin\tau\right]. \qquad (2.25)$$

This function evidently increases unboundedly with increasing variable τ. It should be also emphasized that the functions $A_0(\mathcal{T})$ and $B_0(\mathcal{T})$ are still not determined.

In accordance to the two-scale method, one should get rid of secular terms in solutions to Eqs. (2.18), (2.19), ... As a consequence of this, all the arbitrariness in the asymptotic constructions will be eliminated. So, in order Eq. (2.24) to have a bounded solution (2.25) with respect to the variable τ, it is necessary to satisfy the equalities

$$\frac{dA_0}{d\mathcal{T}} + A_0 = 0, \quad \frac{dB_0}{d\mathcal{T}} + B_0 = 0. \qquad (2.26)$$

From Eqs. (2.26) and the initial conditions (2.23), it follows that $A_0(\mathcal{T}) = Ae^{-\mathcal{T}}$ and $B_0(\mathcal{T}) = 0$. Hence, according to (2.22) and (2.25), we will have

$$x_0 = Ae^{-\mathcal{T}}\cos\tau, \qquad (2.27)$$

$$x_1 = A_1(\mathcal{T})\cos\tau + B_1(\mathcal{T})\sin\tau. \qquad (2.28)$$

Moreover, in view of the initial conditions (2.21), we get

$$A_1(0) = 0, \quad B_1(0) = A. \qquad (2.29)$$

Substituting the expressions (2.27) and (2.28) into the right-hand side of Eq. (2.19), we obtain the following equation for determining the function $x_2(\tau, \mathcal{T})$:

$$\frac{\partial^2 x_2}{\partial\tau^2} + x_2 = 2\left[\left(\frac{dA_1}{d\mathcal{T}} + A_1\right)\sin\tau - \left(\frac{dB_1}{d\mathcal{T}} + B_1 - \frac{1}{2}Ae^{-\mathcal{T}}\right)\cos\tau\right].$$

This equation will have solutions without secular terms, if the following relations are satisfied:

$$\frac{dA_1}{d\mathcal{T}} + A_1 = 0, \quad \frac{dB_1}{d\mathcal{T}} + B_1 = \frac{1}{2}Ae^{-\mathcal{T}}.$$

From here, taking into account the initial conditions (2.29), we find

$$A_1(\mathcal{T}) = 0, \quad B_1(\mathcal{T}) = Ae^{-\mathcal{T}}\big(1 + (\mathcal{T}/2)\big).$$

Thus, the function (2.28) has been determined completely, that is

$$x_1 = Ae^{-\mathcal{T}}\left(\frac{\mathcal{T}}{2} + 1\right)\sin\tau. \tag{2.30}$$

According to the expansion (2.9), formulas (2.27) and (2.30) yield the following two-term asymptotics of the solution to the problem under consideration:

$$x \simeq Ae^{-\mathcal{T}}\cos\tau + \varepsilon Ae^{-\mathcal{T}}\left(\frac{\mathcal{T}}{2} + 1\right)\sin\tau. \tag{2.31}$$

Returning to the real time variable t by use of formulas (2.7), we finally get

$$x \simeq Ae^{-\beta t}\left[\cos\omega_0 t + \frac{\beta}{\omega_0}\left(\frac{1}{2}\beta t + 1\right)\sin\omega_0 t\right]. \tag{2.32}$$

Let us confront the obtained approximate solution with the following exact solution of Eq. (2.5) satisfying the initial condition (2.6):

$$x = Ae^{-\beta t}\left[\cos\sqrt{\omega_0^2 - \beta^2}\,t + \frac{\beta}{\sqrt{\omega_0^2 - \beta^2}}\sin\sqrt{\omega_0^2 - \beta^2}\,t\right]. \tag{2.33}$$

Expanding the expression in the square braces in (2.33) in a power series with respect to the parameter $\varepsilon = \beta/\omega_0$, we receive evidence that the first two terms of the Maclaurin series coincide with the expression in the square braces in formula (2.32).

Continuing the process of construction of the terms of expansion (2.9), one can easily determine the function $x_2(\tau, \mathcal{T})$. According to formula (2.31), the function $x_2(\tau, \mathcal{T})$ will yield the correction of order $O(\varepsilon^2\mathcal{T}^2)$. Hence both the two-term asymptotic representation (2.31) and the asymptotic expansion (2.9) with even more terms turn out to be valid only on a finite interval $0 \le \mathcal{T} \le \mathcal{T}_1$ of the "slow" time variable $\mathcal{T}$, i.e., on the time interval $0 \le t \le (\varepsilon\omega_0)^{-1}\mathcal{T}_1$ (see also [11]).

In contrast to the straightforward expansion according to the perturbation method, the two-scale method allows the ε^{-1}-times extension of the validity interval for the asymptotic solution.

2.3 Krylov — Bogoliubov method

2.3.1 Krylov — Bogoliubov variables

Again, let us consider the equation of quasilinear oscillations

$$\ddot{x} + \omega_0^2 x = \mu f(x, \dot{x}), \tag{3.1}$$

where μ is a small positive parameter.

Following Krylov and Bogoliubov [9], the general solution to Eq. (3.1) will be constructed in the form of expansion

$$x = a\cos\psi + \mu u_1(a,\psi) + \mu^2 u_2(a,\psi) + \dots\,, \tag{3.2}$$

where $u_1(a,\psi)$, $u_2(a,\psi)$, … are 2π-periodic functions of the angle ψ, while a and ψ as functions of time are determined from the system of differential equations

$$\begin{cases} \dot{a} = \mu A_1(a) + \mu^2 A_2(a) + \dots, \\ \dot{\psi} = \omega_0 + \mu B_1(a) + \mu^2 B_2(a) + \dots . \end{cases} \tag{3.3}$$

At that, it is assumed that the Fourier series of the functions $u_1(a,\psi)$, $u_2(a,\psi)$, …, do not contain the first harmonics. In other words, these periodic functions of the phase angle ψ will be determined in such a way that

$$\int\limits_0^{2\pi} u_k(a,\psi) \begin{Bmatrix} \cos\psi \\ \sin\psi \end{Bmatrix} d\psi = 0 \quad (k = 1, 2, \dots). \tag{3.4}$$

Note that from a physical point of view, the conditions (3.4) correspond to the condition that the quantity a coincides with the total amplitude of the first harmonics of oscillations.

Thus, the problem of integration of Eq. (3.1) will reduced to a more simple problem of integration of Eqs. (3.3) with separable variables as soon as the problem of construction of explicit expressions for the functions $u_1(a,\psi)$, $u_2(a,\psi)$, … , $A_1(a)$, $B_1(a)$, $A_2(a)$, $B_2(a)$, … is solved in such a way that the expression (3.2) with a and ψ replaced by the functions of time in accordance with Eqs. (3.3), represents the sought-for solution to the original equation (3.1).

Practically, due to analytical difficulties, only the first several terms in the asymptotic expansions (3.2) and (3.3) can be determined. Leaving only

n terms in these expansions, we will have

$$x = a\cos\psi + \mu u_1(a,\psi) + \mu^2 u_2(a,\psi) + \ldots + \mu^n u_n(a,\psi); \tag{3.5}$$

$$\begin{cases} \dot{a} = \mu A_1(a) + \mu^2 A_2(a) + \ldots + \mu^n A_n(a), \\ \dot{\psi} = \omega_0 + \mu B_1(a) + \mu^2 B_2(a) + \ldots + \mu^n B_n(a). \end{cases} \tag{3.6}$$

It should be underlined that the practical applicability of the perturbation method is determined not by the property of convergence of the series (3.5) and (3.6) as $n \to \infty$, but by their asymptotic properties for a certain fixed n as $\mu \to 0$. It is required that for small values of the parameter μ, the expression (3.5) should give a sufficiently accurate representation of the solution to Eq. (3.1) for a long period of time.

2.3.2 Krylov — Bogoliubov technique

By the chain rule of differentiation we have

$$\frac{d}{dt} = \dot{a}\frac{\partial}{\partial a} + \dot{\psi}\frac{\partial}{\partial \psi},$$

$$\frac{d^2}{dt^2} = \ddot{a}\frac{\partial}{\partial a} + \dot{a}^2\frac{\partial^2}{\partial a^2} + 2\dot{a}\dot{\psi}\frac{\partial^2}{\partial a\partial\psi} + \dot{\psi}^2\frac{\partial^2}{\partial\psi^2} + \ddot{\psi}\frac{\partial}{\partial\psi}.$$

Differentiating the right-hand side of the expansion (3.2), we find

$$\begin{aligned} \frac{dx}{dt} = {} & \dot{a}\left(\cos\psi + \mu\frac{\partial u_1}{\partial a} + \mu^2\frac{\partial u_2}{\partial a} + \ldots\right) \\ & + \dot{\psi}\left(-a\sin\psi + \mu\frac{\partial u_1}{\partial\psi} + \mu^2\frac{\partial u_2}{\partial\psi} + \ldots\right), \end{aligned}$$

$$\begin{aligned} \frac{d^2x}{dt^2} = {} & \ddot{a}\left(\cos\psi + \mu\frac{\partial u_1}{\partial a} + \mu^2\frac{\partial u_2}{\partial a} + \ldots\right) \\ & + \dot{a}^2\left(\mu\frac{\partial^2 u_1}{\partial a^2} + \mu^2\frac{\partial^2 u_2}{\partial a^2} + \ldots\right) \\ & + 2\dot{a}\dot{\psi}\left(-\sin\psi + \mu\frac{\partial^2 u_1}{\partial a\partial\psi} + \mu^2\frac{\partial^2 u_2}{\partial a\partial\psi} + \ldots\right) \\ & + \dot{\psi}^2\left(-a\cos\psi + \mu\frac{\partial^2 u_1}{\partial\psi^2} + \mu^2\frac{\partial^2 u_2}{\partial\psi^2} + \ldots\right) \\ & + \ddot{\psi}\left(-a\sin\psi + \mu\frac{\partial u_1}{\partial\psi} + \mu^2\frac{\partial u_2}{\partial\psi} + \ldots\right). \end{aligned}$$

Tacking into account Eq. (3.3), we derive the following expansions:

$$\dot{a}^2 = (\mu A_1 + \mu^2 A_2 + \ldots)^2 = \mu^2 A_1^2 + \mu^3 \ldots ,$$

$$\begin{aligned}\dot{a}\dot{\psi} &= (\mu A_1 + \mu^2 A_2 + \ldots)(\omega_0 + \mu B_1 + \mu^2 B_2 + \ldots)\\ &= \mu\omega_0 A_1 + \mu^2(\omega_0 A_2 + A_1 B_1) + \mu^3 \ldots ,\end{aligned}$$

$$\begin{aligned}\dot{\psi}^2 &= (\omega_0 + \mu B_1 + \mu^2 B_2 + \ldots)^2\\ &= \omega_0^2 + 2\mu\omega_0 B_1 + \mu^2(B_1^2 + 2\omega_0 B_2) + \mu^3 \ldots ,\end{aligned}$$

$$\begin{aligned}\ddot{a} &= \left(\mu \frac{dA_1}{da} + \mu^2 \frac{dA_2}{da} + \ldots\right)\dot{a}\\ &= \left(\mu \frac{dA_1}{da} + \mu^2 \frac{dA_2}{da} + \ldots\right)(\mu A_1 + \mu^2 A_2 + \ldots)\\ &= \mu^2 A_1 \frac{dA_1}{da} + \mu^3 \ldots ,\end{aligned}$$

$$\ddot{\psi} = \left(\mu \frac{dB_1}{da} + \mu^2 \frac{dB_2}{da} + \ldots\right)(\mu A_1 + \mu^2 A_2 + \ldots) = \mu^2 A_1 \frac{dB_1}{da} + \mu^3 \ldots .$$

Using these formal asymptotic expansions, we obtain

$$\begin{aligned}\frac{dx}{dt} &= -\omega_0 a \sin\psi + \mu\left(A_1 \cos\psi - aB_1 \sin\psi + \omega_0 \frac{\partial u_1}{\partial \psi}\right)\\ &\quad + \mu^2\left(A_2 \cos\psi - aB_2 \sin\psi + A_1 \frac{\partial u_1}{\partial a} + B_1 \frac{\partial u_1}{\partial \psi} + \omega_0 \frac{\partial u_2}{\partial \psi}\right) + \mu^3 \ldots ,\end{aligned}$$

$$\begin{aligned}\frac{d^2x}{dt^2} &= -\omega_0^2 a \cos\psi + \mu\left(-2\omega_0 A_1 \sin\psi - 2\omega_0 a B_1 \cos\psi + \omega_0^2 \frac{\partial^2 u_1}{\partial \psi^2}\right)\\ &\quad + \mu^2\Bigg\{\left(A_1 \frac{dA_1}{da} - aB_1^2 - 2\omega_0 a B_2\right)\cos\psi\\ &\quad - \left(2\omega_0 A_2 + 2A_1 B_1 + A_1 \frac{dB_1}{da} a\right)\sin\psi\\ &\quad + 2\omega_0 A_1 \frac{\partial^2 u_1}{\partial a \partial \psi} + 2\omega_0 B_1 \frac{\partial^2 u_1}{\partial \psi^2} + \omega_0^2 \frac{\partial^2 u_2}{\partial \psi^2}\Bigg\} + \mu^3 \ldots .\end{aligned}$$

So, the left-hand side of Eq. (3.1) can be represented in the form of a power series with respect to parameter μ as

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \mu\left(-2\omega_0 A_1 \sin\psi - 2\omega_0 a B_1 \cos\psi + \omega_0^2 \frac{\partial^2 u_1}{\partial\psi^2} + \omega_0^2 u_1\right)$$
$$+ \mu^2 \left\{ \left(A_1 \frac{dA_1}{da} - aB_1^2 - 2\omega_0 a B_2\right)\cos\psi \right.$$
$$+ \left(2\omega_0 A_2 + 2A_1 B_1 + A_1 \frac{dB_1}{da} a\right)\sin\psi \tag{3.7}$$
$$\left. + 2\omega_0 A_1 \frac{\partial^2 u_1}{\partial a \partial\psi} + 2\omega_0 B_1 \frac{\partial^2 u_1}{\partial\psi^2} + \omega_0^2 \frac{\partial^2 u_2}{\partial\psi^2} + \omega_0^2 u_2 \right\} + \mu^3 \ldots .$$

On the other hand, the right-hand side of Eq. (3.1) can be represented as follows:

$$\mu f\left(x, \frac{dx}{dt}\right) = \mu f(a\cos\psi, -\omega_0 a \sin\psi) + \mu^2 \left\{ u_1 f'_x(a\cos\psi, -\omega_0 a\sin\psi) \right.$$
$$\left. + \left(A_1 \cos\psi - aB_1 \sin\psi + \omega_0 \frac{\partial u_1}{\partial\psi}\right) f'_{\dot{x}}(a\cos\psi, -\omega_0 a \sin\psi) \right\} + \mu^3 \ldots . \tag{3.8}$$

In order the expression (3.2) to satisfy the original equation (3.1) with the accuracy up to the terms of order μ^{n+1}, it is necessary to equate the coefficients standing by same powers of μ in the right-hand sides of the relations (3.7) and (3.8) up to terms of the $n-1$-th order inclusive. As a result, we get

$$\begin{aligned}
\omega_0^2\left(\frac{\partial^2 u_1}{\partial\psi^2} + u_1\right) &= f_0(a,\psi) + 2\omega_0 A_1 \sin\psi + 2\omega_0 a B_1 \cos\psi, \\
\omega_0^2\left(\frac{\partial^2 u_2}{\partial\psi^2} + u_2\right) &= f_1(a,\psi) + 2\omega_0 A_2 \sin\psi + 2\omega_0 a B_2 \cos\psi, \\
&\cdots \\
\omega_0^2\left(\frac{\partial^2 u_n}{\partial\psi^2} + u_n\right) &= f_{n-1}(a,\psi) + 2\omega_0 A_n \sin\psi + 2\omega_0 a B_n \cos\psi,
\end{aligned} \tag{3.9}$$

where, in particular, we used the notation

$$
\begin{aligned}
f_0(a,\psi) &= f(a\cos\psi, -\omega_0 a\sin\psi),\\
f_1(a,\psi) &= u_1 f'_x(a\cos\psi, -\omega_0 a\sin\psi)\\
&\quad+\left(A_1\cos\psi - aB_1\sin\psi + \omega_0\frac{\partial u_1}{\partial\psi}\right) f'_{\dot x}(a\cos\psi, -\omega_0 a\sin\psi)\\
&\quad+\left(aB_1^2 - A_1\frac{dA_1}{da}\right)\cos\psi + \left(2A_1B_1 + A_1 a\frac{dB_1}{da}\right)\sin\psi\\
&\quad- 2\omega_0 A_1\frac{\partial^2 u_1}{\partial a\partial\psi} - 2\omega_0 B_1\frac{\partial^2 u_1}{\partial\psi^2}.
\end{aligned}
$$

It is easy to see that $f_k(a,\psi)$ is a 2π-periodic function of the variable ψ. Moreover, its exact expression is known as soon as the expressions $A_j(a)$, $B_j(a)$, $u_j(a,\psi)$ are determined up to the k-th order inclusive.

In order to determine the functions $u_1(a,\psi)$, $A_1(a)$, and $B_1(a)$ from the first equation of the system (3.9), we consider the Fourier expansions of the functions $f_0(a,\psi)$ and $u_1(a,\psi)$:

$$
\begin{aligned}
f_0(a,\psi) &= \frac{g_0(a)}{2} + \sum_{k=1}^{\infty} g_k^c(a)\cos k\psi + g_k^s(a)\sin k\psi,\\
u_1(a,\psi) &= \frac{v_0(a)}{2} + \sum_{k=1}^{\infty} v_k^c(a)\cos k\psi + v_k^s(a)\sin k\psi.
\end{aligned}
$$

Substituting the given expansions into the first equation of the system (3.9), we obtain

$$
\begin{aligned}
&\frac{\omega_0^2}{2}v_0(a) + \sum_{k=1}^{\infty}\omega_0^2(1-k^2)\big(v_k^c(a)\cos k\psi + v_k^s(a)\sin k\psi\big) =\\
&\quad= \frac{1}{2}g_0(a) + \big(g_1^c(a) + 2\omega_0 aB_1\big)\cos\psi + \big(g_1^s(a) + 2\omega_0 aA_1\big)\sin\psi\\
&\quad+ \sum_{k=2}^{\infty} g_k^c(a)\cos k\psi + g_k^s(a)\sin k\psi.
\end{aligned}
$$

Equating the coefficients standing by same harmonics, we find

$$
g_1^c(a) + 2\omega_0 aB_1 = 0, \quad g_1^s(a) + 2\omega_0 aA_1 = 0, \tag{3.10}
$$

$$v_0(a) = \frac{g_0(a)}{2\omega_0^2}, \quad \begin{Bmatrix} v_k^c(a) \\ v_k^s(a) \end{Bmatrix} = \frac{1}{\omega_0^2(1-k^2)} \begin{Bmatrix} g_k^c(a) \\ g_k^s(a) \end{Bmatrix} \quad (k = 2, 3, \ldots).$$

Moreover, due to the additional conditions (3.4), we have

$$v_1^c(a) = 0, \quad v_1^s(a) = 0.$$

Note that Eqs. (3.10) derived for determining the functions $A_1(a)$ и $B_1(a)$ imply the absence of the first harmonics in the right-hand side of Eq. (3.9). This in turn allows to avoid secular terms in its solution.

Having determined the functions $u_1(a, \psi)$, $A_1(a)$, and $B_1(a)$, we also obtain an explicit expression for the function $f_1(a, \psi)$. Proceeding in such a way, we generate the process of successive construction all the terms of the asymptotic expansions (3.5) and (3.6) from the equations of the system (3.9).

2.3.3 First-order approximation equations

Recall that for $\mu = 0$, Eq. (3.1) admits the solution

$$x = a\cos\psi, \quad \dot{x} = -a\omega_0 \sin\psi, \tag{3.11}$$

where $\psi = \omega_0 t + \theta$. At that, the amplitude a and the phase of oscillations θ are constants.

Formulas (3.11) hold true also in the case $\mu \neq 0$ under the condition that the quantities a and θ are considered as some functions of time. In other words, formulas (3.11) will be considered as a change of variables, where the amplitude a and the total phase of oscillations ψ are regarded as new sought-for functions (*Krylov – Bogoliubov variables*).

In order to formulate the equations for determining the functions a and ψ, let us differentiate the both sides of the first formula (3.11). We have

$$\dot{x} = \dot{a}\cos\psi - a\dot{\psi}\sin\psi.$$

From here taking into account the second relation (3.11), it follows that

$$\dot{a}\cos\psi - a\dot{\psi}\sin\psi = -a\omega_0\sin\psi. \tag{3.12}$$

Differentiating now the both sides of the second formula (3.11), we get

$$\ddot{x} = -\dot{a}\omega_0\sin\psi - a\omega_0\dot{\psi}\cos\psi. \tag{3.13}$$

Substituting now into Eqs. (3.1) the expressions for x, $\dot{x}$, and $\ddot{x}$ according to formulas (3.11), and (3.13), we find

$$-\dot{a}\omega_0 \sin\psi - a\omega_0\dot{\psi}\cos\psi = -a\omega_0^2\cos\psi + \mu f(a\cos\psi, -\omega_0 a\sin\psi). \quad (3.14)$$

Resolving the system of the two equations (3.12) and (3.14) for the derivatives $\dot{a}$ and $\dot{\psi}$, we obtain the following system:

$$\begin{cases} \dot{a} = -\dfrac{\mu}{\omega_0} f(a\cos\psi, -\omega_0 a\sin\psi)\sin\psi, \\ \dot{\psi} = \omega_0 - \dfrac{\mu}{a\omega_0} f(a\cos\psi, -\omega_0 a\sin\psi)\cos\psi. \end{cases} \quad (3.15)$$

Thus, instead of one differential equation of the second order (3.1) with respect to the variable x, we obtain the system of two first order differential equations (3.15) with respect to variables a and ψ.

Finally, introducing a new variable θ (phase of oscillations) instead of the total phase of oscillation ψ according to the formula

$$\psi = \omega_0 t + \theta,$$

we will have

$$\begin{cases} \dot{a} = -\dfrac{\mu}{\omega_0} f(a\cos\psi, -\omega_0 a\sin\psi)\sin\psi, \\ \dot{\theta} = -\dfrac{\mu}{a\omega_0} f(a\cos\psi, -\omega_0 a\sin\psi)\cos\psi. \end{cases} \quad (3.16)$$

Note that the right-hand sides of Eqs. (3.16) posses the period $2\pi/\omega_0$ with respect to the variable t. Moreover, the derivatives $\dot{a}$ and $\dot{\theta}$ are proportional to small parameter μ, thus, the amplitude a and the phase of oscillations θ will be slowly varying functions of time.

Let us expand the right-hand sides of Eqs. (3.16) into Fourier series as follows:

$$-\frac{1}{\omega_0} f(a\cos\psi, -\omega_0 a\sin\psi)\sin\psi =$$

$$= \frac{h_{10}(a)}{2} + \sum_{k=1}^{\infty} h_{1k}^c(a)\cos k\psi + h_{1k}^s(a)\sin k\psi,$$

$$-\frac{1}{a\omega_0} f(a\cos\psi, -\omega_0 a\sin\psi)\cos\psi =$$

$$= \frac{h_{20}(a)}{2} + \sum_{k=1}^{\infty} h_{2k}^{c}(a) \cos k\psi + h_{2k}^{s}(a) \sin k\psi.$$

Taking into consideration the fact that a and θ are slowly varying quantities, we represent them as superpositions of slowly varying quantities $\overline{a}$ and $\overline{\theta}$ and small oscillating terms. In the first-order approximation, we put

$$a = \overline{a}, \quad \theta = \overline{\theta} \quad (\overline{\psi} = \omega_0 t + \overline{\theta}).$$

Correspondingly, we obtain

$$\begin{cases} \dfrac{d\overline{a}}{dt} = \dfrac{\mu}{2} h_{10}(\overline{a}) + \text{small oscillating terms,} \\ \dfrac{d\overline{\theta}}{dt} = \dfrac{\mu}{2} h_{20}(\overline{a}) + \text{small oscillating terms.} \end{cases}$$

Assuming that the distinguished small oscillating terms generate only small oscillations of the quantities a and θ about their first approximations $\overline{a}$, $\overline{\theta}$ and do not influence the systematical behavior of a and θ, we arrive at the first-order approximation equations

$$\begin{aligned} \frac{d\overline{a}}{dt} &= \frac{\mu}{2} h_{10}(\overline{a}) = \mu \underset{t}{M} \left\{ -\frac{1}{\omega_0} f(a\cos\psi, -\omega_0 a \sin\psi) \sin\psi \right\}, \\ \frac{d\overline{\theta}}{dt} &= \frac{\mu}{2} h_{20}(\overline{a}) = \mu \underset{t}{M} \left\{ -\frac{1}{a\omega_0} f(a\cos\psi, -\omega_0 a \sin\psi) \cos\psi \right\}. \end{aligned} \tag{3.17}$$

Here we introduced the averaging operator under the constant quantities $\overline{a}$ and $\overline{\theta}$ with respect to the explicitly occurring time variable t.

It is not hard to see that Eqs. (3.17) obtained for determining the functions $\overline{a}$ and $\overline{\theta}$ coincide with the previously derived first-order approximation equations (see formulas (3.3) and (3.10)). In fact, averaging Eqs. (3.15), we obtain

$$\begin{cases} \dot{a} = -\dfrac{\mu}{2\pi\omega_0} \displaystyle\int_0^{2\pi} f(a\cos\psi, -\omega_0 a \sin\psi) \sin\psi \, d\psi, \\ \dot{\psi} = \omega_0 - \dfrac{\mu}{2\pi\omega_0 a} \displaystyle\int_0^{2\pi} f(a\cos\psi, -\omega_0 a \sin\psi) \cos\psi \, d\psi. \end{cases} \tag{3.18}$$

Finally, note that the first-order approximation equations derived by the Krylov—Bogoliubov method coincide with the approximate equations which can be derived by the Van der Pol perturbation method.

2.3.4 Stationary regimes

The system (3.18) allows to find possible stationary (self-oscillation) regimes when the amplitude of oscillations remains constant.

Setting $da/dt = 0$, we find that the stationary amplitude should be a root of the equation

$$\int_0^{2\pi} f(a\cos u, -\omega_0 a \sin u)\sin u\, du = 0. \tag{3.19}$$

Note that Eq. (3.19) may not have real solutions at all. Such a situation means that there is no possibility for realization of the stationary oscillations in the mechanical system under consideration. On the other hand, Eq. (3.19) may have several solutions or be identically satisfied (for instance, when the system is conservative).

Note that in interpreting the results of the study of the so-called truncated equations (3.18) it is necessary to have in mind that these results reflect correctly the properties of the original equations (3.1) only under the condition that the parameter μ is sufficiently small (see [4], Ch. 9, § 3 and § 6).

2.3.5 Equivalent linearization of quasilinear oscillating systems

Consider the governing differential equation of a quasilinear oscillating system

$$m\ddot{x} + kx = \mu m f(x, \dot{x}), \tag{3.20}$$

where m and k are positive constants, μ is a small dimensionless parameter.

By the Krylov — Bogoliubov method, the solution to Eq. (3.20) in the first approximation can be represented in the form

$$x = a\cos\psi.$$

At that, the amplitude a and the total phase of oscillations ψ should satisfy Eqs. (3.18) with

$$\omega_0^2 = \frac{k}{m}.$$

Let us show that the oscillations of the system under consideration are approximately equivalent (with the accuracy up to the terms that are neglected in deriving the first-order approximation equation (3.18)) to those of a certain linear oscillation system, which is described by the equation

$$m\ddot{x} + b_e(a)\dot{x} + k_e(a)x = 0 \tag{3.21}$$

with the stiffness coefficient $k_e(a)$ and the damping coefficient $b_e(a)$.

It is well known that Eq. (3.21) is equivalent to the system

$$\dot{x} = y, \quad \dot{y} = -\frac{b_e}{m}y - \omega_e^2 x, \tag{3.22}$$

where we used the notation

$$\omega_e^2 = \frac{k_e}{m}. \tag{3.23}$$

Let us now introduce the Krylov—Bogoliubov variables

$$x = a\cos\psi, \quad y = -a\omega_e\sin\psi, \tag{3.24}$$

where the quantity ω_e as well as the quantity b_e are regarded as constant. Substituting the expressions (3.24) into Eqs. (3.22), we obtain

$$\dot{a}\cos\psi - a\dot{\psi}\sin\psi = -a\omega_e\sin\psi,$$

$$-\dot{a}\omega_e\sin\psi - a\omega_e\dot{\psi}\cos\psi = \frac{b_e}{m}a\omega_e\sin\psi - \omega_e^2 a\cos\psi.$$

Resolving the given system of equations for the derivatives $\dot{a}$ и $\dot{\psi}$, we find

$$\dot{a} = -\frac{b_e}{m}a\sin^2\psi, \quad \dot{\psi} = \omega_e - \frac{b_e}{m}\omega_e\sin\psi\cos\psi. \tag{3.25}$$

Under the assumption that the value of the damping coefficient b_e is relatively small, we make use of the averaging procedure for the equations of system (3.25). As a result, we obtain the following first-order approximation system:

$$\dot{a} = -\frac{b_e}{2m}a, \quad \dot{\psi} = \omega_e. \tag{3.26}$$

Equating the right-hand sides of Eqs. (3.26) and (3.18), we get

$$\omega_e = \omega_0 - \frac{\mu}{2\pi\omega_0 a}\int_0^{2\pi} f(a\cos\psi, -\omega_0 a\sin\psi)\cos\psi\, d\psi, \tag{3.27}$$

$$b_e = \frac{\mu m}{\pi \omega_0 a} \int_0^{2\pi} f(a\cos\psi, -\omega_0 a \sin\psi) \sin\psi \, d\psi. \tag{3.28}$$

Finally, squaring the both sides of Eq. (3.27) and neglecting small terms of the order μ^2, we find

$$k_e = k - \frac{\mu m}{\pi a} \int_0^{2\pi} f(a\cos\psi, -\omega_0 a \sin\psi) \cos\psi \, d\psi. \tag{3.29}$$

Here we used formula (3.23).

Thus, in the first approximation, the oscillations of the original quasilinear system are equivalent (with the accuracy up to small terms of order μ^2) to the oscillations of a certain linear mechanical system with the stiffness coefficient $k_e(a)$ and the damning coefficient $b_e(a)$ depending on the amplitude of oscillations a according to formulas (3.29) and (3.28).

2.4 Method of matched asymptotic expansions

2.4.1 Oscillator with small mass. System with 1/2 degree of freedom

Let us consider a damped linear oscillator described by the differential equation

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{4.1}$$

with the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = v_0. \tag{4.2}$$

We will study the motion of a body of small mass in a highly damped medium under the action of a linearly elastic spring. Neglecting the first term in Eq. (4.1) we obtain the *limit* differential equation of the first order

$$b\dot{x} + kx = 0. \tag{4.3}$$

In this way, we arrive at the so-called [4] system with $^1/_2$ degree of freedom. The phase space of this system is one-dimensional. Correspondingly, for unique determination of the behavior of such a system it is necessary to fix only one quantity (coordinate x) at the initial moment of time instead of the two quantities x and $\dot{x}$, which are necessary for unique determination of the behavior of a system with one degree of freedom. Let us underline that assuming that Eq. (4.3) holds true at every moment of time, we will have $\dot{x}(0) = -(k/b)x(0)$ for $t = 0$.

Thus, the initial value of the velocity v_0 cannot be taken arbitrarily irrespective of the initial value of the coordinate x_0.

The solution to Eq. (4.3) subjected to the first initial condition (4.2) has the form

$$x = x_0 \exp\left\{-\frac{k}{b}t\right\}. \tag{4.4}$$

It is evident that the position $x_0 = 0$ coincides with the equilibrium position. For any other initial conditions, according to (4.4), the oscillator without mass performs damped aperiodic motion, which converges to the equilibrium position as $t \to +\infty$.

2.4.2 Outer asymptotic expansion

Let us introduce a dimensionless independent time variable by the formula

$$\tau = \frac{k}{b}t. \tag{4.5}$$

According to (4.5), we rewrite Eq. (4.1) as follows:

$$\varepsilon\frac{d^2x}{d\tau^2} + \frac{dx}{d\tau} + x = 0. \tag{4.6}$$

Here we introduced the notation

$$\varepsilon = \frac{mk}{b^2}. \tag{4.7}$$

Correspondingly, the initial conditions (4.2) take the form

$$x(0) = x_0, \quad \frac{dx}{d\tau}(0) = \frac{bv_0}{k}. \tag{4.8}$$

We will study the behavior of the solution to the problem (4.6), (4.8) for small values of the dimensionless positive parameter ε. Let us try as before to find the solution to Eq. (4.6) in the form of expansion

$$x = \xi_0(\tau) + \varepsilon\xi_1(\tau) + \dots . \tag{4.9}$$

Substituting the series (4.9) into Eq. (4.6), we get

$$\frac{d\xi_0}{d\tau} + \xi_0 = 0, \tag{4.10}$$

$$\frac{d\xi_k}{d\tau} + \xi_k = -\frac{d^2\xi_{k-1}}{d\tau^2} \quad (k = 1, 2, \ldots). \tag{4.11}$$

The general solution of Eq. (4.10) is

$$\xi_0(\tau) = C_0 e^{-\tau}. \tag{4.12}$$

The substitution of the expression (4.12) into Eq. (4.11), $k = 1$, leads to its general solution

$$\xi_1(\tau) = -C_0 \tau e^{-\tau} + C_1 e^{-\tau}. \tag{4.13}$$

For determining the integration constants C_0 and C_1 it is necessary to turn to the initial conditions (4.8). Substituting the expansion (4.9) into the relations (4.8), we obtain formally two initial conditions for each of Eqs. (4.10) and (4.11). This fact immediately leads to the contradiction mentioned above.

Therefore, at the initial moments of time, the solution to Eq. (4.6) should be sought-for in the form of a cognizably different asymptotic expansion from (4.9), in order to have a possibility to satisfy both initial conditions (4.8). In other words, not far from the initial time moment $\tau = 0$, there arises the phenomenon of a boundary layer.

2.4.3 Inner asymptotic expansion

Let us introduce the so-called "fast" variable

$$\mathcal{T} = \frac{\tau}{\varepsilon}. \tag{4.14}$$

Making use of the change of the independent variable (4.14) in Eq. (4.6) and the initial conditions (4.8), we get

$$\frac{1}{\varepsilon}\left(\frac{d^2x}{d\mathcal{T}^2} + \frac{dx}{d\mathcal{T}}\right) + x = 0; \tag{4.15}$$

$$x(0) = x_0, \quad \frac{1}{\varepsilon}\frac{dx}{d\mathcal{T}}(0) = \frac{bv_0}{k}. \tag{4.16}$$

At the initial moments of time, the solution to Eq. (4.6) will be represented in the form of expansion

$$x = X_0(\mathcal{T}) + \varepsilon X_1(\mathcal{T}) + \dots . \tag{4.17}$$

Substituting the expansion (4.17) into Eq. (4.15), we arrive at the following system of equations:

$$\frac{d^2 X_0}{d\mathcal{T}^2} + \frac{dX_0}{d\mathcal{T}} = 0, \tag{4.18}$$

$$\frac{d^2 X_l}{d\mathcal{T}^2} + \frac{dX_l}{d\mathcal{T}} = -X_{l-1} \quad (l = 1, 2, \dots). \tag{4.19}$$

The substitution of the expansion (4.17) into the initial conditions (4.16) yields

$$X_0(0) = x_0, \quad \frac{dX_0}{d\mathcal{T}}(0) = 0; \tag{4.20}$$

$$X_1(0) = 0, \quad \frac{dX_1}{d\mathcal{T}}(0) = \frac{bv_0}{k}; \tag{4.21}$$

$$X_l(0) = 0, \quad \frac{dX_l}{d\mathcal{T}}(0) = 0 \quad (l = 2, 3, \dots). \tag{4.22}$$

The solution of the recurrence differential equations (4.18), (4.19) subjected to the initial conditions (4.20) – (4.22) can be constructed step by step. So, the general solution of Eq. (4.18) has the form

$$X_0(\mathcal{T}) = B_0 + D_0 e^{-\mathcal{T}}, \quad \frac{dX_0}{d\mathcal{T}}(\mathcal{T}) = -D_0 e^{-\mathcal{T}}. \tag{4.23}$$

From the initial conditions (4.20), we find $D_0 = 0$ and $B_0 = x_0$, thereby we finally obtain

$$X_0(0) = x_0. \tag{4.24}$$

Substituting the expansion (4.24) into Eq. (4.19), $l = 1$, we find its general solution in the form

$$X_1(\mathcal{T}) = -x_0 \mathcal{T} + B_1 + D_1 e^{-\mathcal{T}}, \quad \frac{dX_1}{d\mathcal{T}}(\mathcal{T}) = -x_0 - D_1 e^{-\mathcal{T}}.$$

Taking now into consideration the initial conditions (4.21), we get

$$X_1(\mathcal{T}) = -x_0 \mathcal{T} + \left(x_0 + \frac{bv_0}{k}\right)\left(1 - e^{-\mathcal{T}}\right). \tag{4.25}$$

It is clear that proceeding this way, one can find the terms of any fixed order in the expansion (4.17).

Thus, the solution to Eq. (4.6) is represented by the following two expansions: the *inner* asymptotic expansion (4.17), which is valid not far from the initial moment of time, and the *outer* asymptotic expansion (4.9), describing the behavior of the mechanical system under consideration after elapsing some short period of time. The terms of the inner expansion are completely determined from the system of equations (4.18), (4.19) with the initial conditions (4.20) – (4.22), while the terms of the outer expansion contain some arbitrariness. Namely, the first terms of the expansion (4.9) are determined up to the constants C_0 and C_1.

2.4.4 Matching of asymptotic expansions

The indicated arbitrariness in the constructions of the terms of outer asymptotic expansion can be eliminated by *asymptotic matching* with the inner asymptotic expansion. Roughly speaking [11], the idea of matching consists in the following: as $\tau \to 0$, the outer expansion should behave in the same way as the inner expansion does when $\mathcal{T} \to \infty$.

Let us perform the asymptotic matching of the two-term asymptotic representations. So, as $\mathcal{T} \to \infty$, we have

$$X_0(\mathcal{T}) + \varepsilon X_1(\mathcal{T}) = x_0 + \varepsilon \left(-x_0 \mathcal{T} + x_0 + \frac{bv_0}{k} \right) + \ldots . \tag{4.26}$$

Here the omission points denote exponentially small terms, that is the terms of order $e^{-\mathcal{T}}$ as $\mathcal{T} \to \infty$.

On the other hand, as $\tau \to 0$, using Maclaurin's series for the exponent, we find

$$\xi_0(\tau) = C_0 \left(1 - \tau + \frac{\tau^2}{2!} - \frac{\tau^3}{3!} + \ldots \right),$$

$$\xi_1(\tau) = -C_0 \tau \left(1 - \tau + \frac{\tau^2}{2!} - \ldots \right) + C_1 \left(1 - \tau + \frac{\tau^2}{2!} - \frac{\tau^3}{3!} + \ldots \right).$$

Introducing now the fast variable (4.14), we get

$$\xi_0(\varepsilon \mathcal{T}) = C_0 \bigl[1 - \varepsilon \mathcal{T} + O(\varepsilon^2 \mathcal{T}^2) \bigr], \quad \varepsilon \mathcal{T} \to 0.$$

Correspondingly, we will have

$$\xi_0(\varepsilon \mathcal{T}) + \varepsilon \xi_1(\varepsilon \mathcal{T}) = C_0 - \varepsilon C_0 \mathcal{T} + \varepsilon C_1 + O(\varepsilon^2 \mathcal{T}^2), \quad \varepsilon \mathcal{T} \to 0. \tag{4.27}$$

Let us show that there is a domain where the both asymptotic expansions hold true. This is the so-called asymptotic matching region, where the inner and outer asymptotic expansions are matched. Indeed, for the values of τ of order ε^{σ} $(\sigma > 0)$, or, that is the same, for the values of $\mathcal{T}$ of order $\varepsilon^{\sigma-1}$, the terms neglected while developing the relations (4.26) and (4.27) turn out to be small compared with the used terms, if we fix the value of σ in the limits $\sigma - 1 < 0$ and $2\sigma > 1$, that is $0.5 < \sigma < 1$.

Thus, in the asymptotic matching region $c_1\varepsilon^{\sigma} < \tau < c_2\varepsilon^{\sigma}$, where $0 < c_1 < c_2$ are fixed constants and $\sigma \in (0.5; 1)$, the following asymptotic relation holds true:

$$\xi_0(\tau) + \varepsilon\xi_1(\tau) - \left[X_0(\mathcal{T}) + \varepsilon X_1(\mathcal{T})\right] = O(\varepsilon^{2\sigma}), \quad \varepsilon \to 0,$$

if only the following identities are satisfied:

$$C_0 = x_0, \quad C_1 = x_0 + \frac{bv_0}{k}.$$

Consequently, in view of (4.12) and (4.13), the formula

$$x \simeq x_0 e^{-\tau} + \varepsilon\left(x_0(1-\tau) + \frac{bv_0}{k}\right)e^{-\tau} \tag{4.28}$$

gives the two-term outer asymptotic representation.

2.4.5 Uniformly suitable asymptotic representation

With the help of the outer asymptotic representation (4.28) and the two-term inner asymptotic representation

$$x \simeq x_0 + \varepsilon\left\{-x_0\mathcal{T} + \left(x_0 + \frac{bv_0}{k}\right)(1 - e^{-\mathcal{T}})\right\} \tag{4.29}$$

one can construct *uniformly suitable* asymptotic representation (see, for example, [11]).

Thus, it is not hard to see that in the inner (4.29) and outer (4.28) asymptotic representations, one can distinguish common terms that were employed in asymptotic matching. So, if we add the expansions (4.28), (4.29) and after that we subtract these common terms from the obtained sum, then we obtain the uniformly suitable asymptotic representation on the whole interval $0 \le t \le \infty$. Following Cole [11], we subtract the common terms

$$C_0 - C_0\varepsilon\mathcal{T} + \varepsilon C_1 = x_0 + \varepsilon\left(-x_0\mathcal{T} + x_0 + \frac{bv_0}{k}\right)$$

from the inner asymptotic expansion in order the depending on the fast variable $\mathcal{T}$ part of the uniformly suitable representation decays exponentially. As a result, we obtain

$$x \simeq x_0 e^{-\tau} + \varepsilon \left(x_0(1-\tau) + \frac{bv_0}{k} \right) e^{-\tau} - \varepsilon \left(x_0 + \frac{bv_0}{k} \right) e^{-\mathcal{T}}. \qquad (4.30)$$

Finally, note that the first term of the uniformly suitable asymptotic representation (4.30) coincides with (4.4).

Chapter 3

Asymptotic methods in heat-conduction theory

3.1 Introduction. Heat-conduction problems

3.1.1 Heat-conduction equation

The process of heat conduction in a solid can be characterized by the distribution of temperature T as a function of the coordinates x_1, x_2, x_3 and the time t. If the temperature distribution is non uniform, then conduction heat flows arise from temperature differences.

By Fourier's law, the heat quantity transferred across the area dS with the normal $\mathbf{n}$ the center at the point $P(x_1, x_2, x_3)$ during the time interval $(t, t + \Delta t)$ in an isotropic solid is determined by the formula

$$dQ = -\varkappa \frac{\partial T}{\partial n}(\mathbf{x}, t)\, dS\, dt.$$

Here, $\varkappa$ is the *thermal conductivity*, $\partial/\partial n$ is the normal derivative with respect to the area-element dS in the direction of the heat flux, i. e.,

$$\frac{\partial T}{\partial n} = \frac{\partial T}{\partial x_1}\cos(\widehat{n, x_1}) + \frac{\partial T}{\partial x_2}\cos(\widehat{n, x_2}) + \frac{\partial T}{\partial x_3}\cos(\widehat{n, x_3}) \equiv \operatorname{grad} T \cdot \mathbf{n}.$$

Thus, the specific heat flux defined as the heat flux per unit area and unit time is given by

$$\mathbf{q} = -\varkappa \operatorname{grad} T.$$

Let $w(\mathbf{x}, t)$ be the heat generation density due to self heating. Then, considering the heat balance in the elementary volume of the solid (see, for example, [21]), we arrive the differential equation

$$\operatorname{div} \mathbf{q} + c\rho \frac{\partial T}{\partial t} = w. \tag{1.1}$$

Here, c is the specific heat capacity, ρ is the material density,

$$-\operatorname{div} \mathbf{q} = \frac{\partial T}{\partial x_1}\Big(\varkappa \frac{\partial T}{\partial x_1}\Big) + \frac{\partial T}{\partial x_2}\Big(\varkappa \frac{\partial T}{\partial x_2}\Big) + \frac{\partial T}{\partial x_3}\Big(\varkappa \frac{\partial T}{\partial x_3}\Big).$$

If the solid is homogeneous, then the thermal conductivity $\varkappa$ does not depend on the coordinates, and the heat equation (1.1) is usually written in the form

$$-\Delta T + \frac{c\rho}{\varkappa}\frac{\partial T}{\partial t} = \frac{w}{\varkappa}, \tag{1.2}$$

where the coefficient $\dfrac{\varkappa}{c\rho}$ is called thermal diffusivity, Δ is the Laplace operator that is

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

According to the classification of partial differential equations, Eq. (1.2) is of the parabolic type. An important feature of this equation is the asymmetry in the behavior of its solutions with respect to time. In other words, Eq. (1.2) predetermines the time directivity of the heat-conduction process distinguishing past and future.

If the heat-conduction process is stationary, the function $T(\mathbf{x})$ satisfies the Poisson equation

$$-\Delta T = \frac{w}{\varkappa}, \tag{1.3}$$

where the intensity $w(\mathbf{x})$ of heat sources does not depend on the time.

In the absence of the heat sources, we will have the Laplace equation

$$\Delta T = 0. \tag{1.4}$$

Eqs. (1.3) and (1.4) are of the elliptic type.

3.1.2 Formulation of the boundary value problems

Recall that for extracting a unique solution of the heat equation (1.1), it is necessary to impose initial and boundary conditions.

A typical initial condition consists in prescribing a continuous distribution of the temperature in the solid Ω at the moment of time t_0, which has been chosen as the origin of time t, i. e.,

$$T(P, t_0) = T_0(P), \quad P \in \Omega. \tag{1.5}$$

The boundary conditions may be different depending on the temperature regime on the surface Γ of the body Ω. There are the following three main types of boundary conditions.

Boundary condition of the first type. On the surface Γ, a prescribed temperature is maintained, i. e.,

$$T(P, t) = T_1(P, t), \quad P \in \Gamma. \tag{1.6}$$

The boundary value problem with the condition (1.6) is called the *Dirichlet problem.*

Boundary condition of the second type. On the surface Γ, a prescribed heat flux is given, i. e.,

$$\varkappa \frac{\partial T}{\partial n}(P, t) = q_0(P, t), \quad P \in \Gamma. \tag{1.7}$$

The boundary value problem with the condition (1.7) is called the *Neumann problem.*

Boundary condition of the third type. It is assumed that the heat flux across the surface Γ is proportional to the difference between the temperature of the body and the temperature of environment, i. e.,

$$\varkappa \frac{\partial T}{\partial n} + \alpha (T - T_c) = 0. \tag{1.8}$$

The boundary condition (1.8) corresponds to the heat exchange according to Newton's law.

3.2 Homogenization of the heat-conduction process in layered media

3.2.1 Application of the multiple scale method

Following Bakhvalov and Panasenko [8], we consider the differential equation describing a stationary temperature field in a non-homogeneous rod with periodic structure

$$\frac{d}{dx}\left(\varkappa_\varepsilon(x)\frac{du}{dx}\right) - f(x) = 0, \quad x \in (0,1). \tag{2.1}$$

Here, $\varkappa_\varepsilon(x)$ is the variable thermal conductivity, u is the temperature distribution, ε is a small positive parameter, and the length of the rod is assumed to be equal to unity. Eq. (2.1) will be subjected to the following boundary conditions:

$$u(0) = g_0, \quad u(1) = g_1. \tag{2.2}$$

We assume that the one-dimensional medium with periodic structure is comprised by periodically repeated elements, called *cells*. We put

$$\varepsilon = \frac{1}{n}, \tag{2.3}$$

where n is a large natural number, and assume that the function $\varkappa_\varepsilon(x)$ is periodic with the period ε. Then, we will have

$$\varkappa_\varepsilon(x) = \varkappa\left(\frac{x}{\varepsilon}\right), \tag{2.4}$$

where $\varkappa_\varepsilon(\xi)$ is a periodic function with the period 1. It is natural to assume that $\varkappa_\varepsilon(\xi) > 0$ for $\xi \in [0,1]$. For the simplicity sake, we also assume that the function $\varkappa_\varepsilon(\xi)$ is differentiable.

Applying the multiple scale method, we will construct the solution to the problem (2.1), (2.2) in the form

$$u = u_0(x,\xi) + \varepsilon u_1(x,\xi) + \varepsilon^2 u_2(x,\xi) + \dots . \tag{2.5}$$

Here along with the so-called *slow* (or *macroscopic*) variable x, we introduced the *fast* (or *microscopic*) variable

$$\xi = \frac{x}{\varepsilon}. \tag{2.6}$$

We assume that the functions $u_i(x,\xi)$ are periodic with respect to the variable ξ with the period 1.

Let us substitute the expansion (2.5) into Eq. (2.1). Taking into account the expression (2.4) and applying the chain rule of differentiation

$$\frac{d}{dx}F\Big(x,\frac{x}{\varepsilon}\Big) = \Big(\frac{\partial F(x,\xi)}{\partial x} + \varepsilon^{-1}\frac{\partial F(x,\xi)}{\partial \xi}\Big)\Big|_{\xi=\varepsilon^{-1}x},$$

we obtain after rearrangement

$$\begin{aligned}\Bigg(&\varepsilon^{-2}\frac{\partial}{\partial\xi}\Big(\varkappa(\xi)\frac{\partial u_0(x,\xi)}{\partial\xi}\Big) + \varepsilon^{-1}\Big[\frac{\partial}{\partial\xi}\Big(\varkappa(\xi)\frac{\partial u_1(x,\xi)}{\partial\xi}\Big)\\ &\quad + \frac{\partial}{\partial\xi}\Big(\varkappa(\xi)\frac{\partial u_0(x,\xi)}{\partial x}\Big) + \frac{\partial}{\partial x}\Big(\varkappa(\xi)\frac{\partial u_0(x,\xi)}{\partial\xi}\Big)\Big]\\ &+ \varepsilon^{0}\Big[\frac{\partial}{\partial\xi}\Big(\varkappa(\xi)\frac{\partial u_2(x,\xi)}{\partial\xi}\Big) + \frac{\partial}{\partial\xi}\Big(\varkappa(\xi)\frac{\partial u_1(x,\xi)}{\partial x}\Big) + \frac{\partial}{\partial x}\Big(\varkappa(\xi)\frac{\partial u_1(x,\xi)}{\partial\xi}\Big)\\ &\quad + \frac{\partial}{\partial x}\Big(\varkappa(\xi)\frac{\partial u_0(x,\xi)}{\partial x}\Big) - f(x)\Big] + \ldots\Bigg)\Bigg|_{\xi=\varepsilon^{-1}x} = 0. \qquad (2.7)\end{aligned}$$

Equating the coefficients of successive powers of ε to zero, we get

$$\begin{aligned}&\frac{\partial}{\partial\xi}\Big(\varkappa(\xi)\frac{\partial u_0(x,\xi)}{\partial\xi}\Big) = 0,\\ &\frac{\partial}{\partial\xi}\Big(\varkappa(\xi)\frac{\partial u_1(x,\xi)}{\partial\xi}\Big) + \frac{\partial}{\partial\xi}\Big(\varkappa(\xi)\frac{\partial u_0(x,\xi)}{\partial x}\Big) + \frac{\partial}{\partial x}\Big(\varkappa(\xi)\frac{\partial u_0(x,\xi)}{\partial\xi}\Big) = 0,\\ &\frac{\partial}{\partial\xi}\Big(\varkappa(\xi)\frac{\partial u_2(x,\xi)}{\partial\xi}\Big) + \frac{\partial}{\partial\xi}\Big(\varkappa(\xi)\frac{\partial u_1(x,\xi)}{\partial x}\Big) \qquad (2.8)\\ &\quad + \frac{\partial}{\partial x}\Big(\varkappa(\xi)\frac{\partial u_1(x,\xi)}{\partial\xi}\Big) + \frac{\partial}{\partial x}\Big(\varkappa(\xi)\frac{\partial u_0(x,\xi)}{\partial x}\Big) - f(x) = 0.\end{aligned}$$

Assuming that the variables x and ξ are independent, the relations $(2.8)_i$ for $i = 0,1,2$ will be regarded as a recurrence series of linear differential equations for the functions $u_i(x,\xi)$ with respect to the variable ξ, considering x as a parameter.

3.2.2 Application of the homogenization method

Let us introduce the *period-average*

$$\langle F(x,\xi)\rangle = \int_0^1 F(x,\xi)\,d\xi. \tag{2.9}$$

We underline that in the last integral, the variables x and ξ are regarded as independent.

From Eq. $(2.8)_0$, it follows that the function $\varkappa(\xi)\partial u_0(x,\xi)/\partial\xi$ does not depend on the fast variable ξ, i.e.,

$$\varkappa(\xi)\frac{\partial u_0(x,\xi)}{\partial\xi} = C_0(x).$$

From here we find

$$\frac{\partial u_0(x,\xi)}{\partial\xi} = \frac{C_0(x)}{\varkappa(\xi)}. \tag{2.10}$$

Now, we apply the averaging operator (2.9) to the both sides of Eq. (2.10). Taking into consideration the periodicity of the function $u_0(x,\xi)$ in ξ, we will have

$$\left\langle\frac{\partial u_0(x,\xi)}{\partial\xi}\right\rangle = \int_0^1 \frac{\partial u_0(x,\xi)}{\partial\xi}\,d\xi = u_0(x,\xi)\Big|_{\xi=0}^{\xi=1} = 0.$$

Thus, as a result of homogenization of Eq. (2.10), we get

$$0 = \left\langle\varkappa(\xi)^{-1}\right\rangle C_0(x).$$

From here it follows that $C_0(x) = 0$, and, consequently, according to Eq. (2.10), the function $u_0(x,\xi)$ does not depend on ξ, i.e.,

$$u_0(x,\xi) = v_0(x). \tag{2.11}$$

Further, in view of (2.11), Eq. $(2.8)_1$ takes the form

$$\frac{\partial}{\partial\xi}\left(\varkappa(\xi)\left(\frac{\partial u_1(x,\xi)}{\partial\xi} + \frac{dv_0}{dx}\right)\right) = 0.$$

From here, by the same argumentation, we derive

$$\varkappa(\xi)\left(\frac{\partial u_1(x,\xi)}{\partial\xi} + \frac{dv_0}{dx}\right) = C_1(x),$$

and, consequently,

$$\frac{\partial u_1(x,\xi)}{\partial \xi} + \frac{dv_0}{dx} = \frac{C_1(x)}{\varkappa(\xi)}. \tag{2.12}$$

Averaging Eq. (2.12), we obtain

$$\frac{dv_0}{dx} = \left\langle \varkappa(\xi)^{-1} \right\rangle C_1(x),$$

from where we find

$$C_1(x) = \widehat{\varkappa}\frac{dv_0}{dx}. \tag{2.13}$$

Here we introduced the notation

$$\widehat{\varkappa} = \left\langle \varkappa(\xi)^{-1} \right\rangle^{-1}. \tag{2.14}$$

Note that the quantity (2.14) is the geometric mean of the function $\varkappa(\xi)$.

Substituting the expression (2.13) into Eq. (2.12), we find

$$\frac{\partial u_1(x,\xi)}{\partial \xi} = \left(\frac{\widehat{\varkappa}}{\varkappa(\xi)} - 1\right)\frac{dv_0}{dx}. \tag{2.15}$$

Integrating this equation, we obtain

$$u_1(x,\xi) = N_1(\xi)\frac{dv_0}{dx} + v_1(x),$$

where $v_1(x)$ is an arbitrary function,

$$N_1(\xi) = \int\limits_0^\xi \left(\frac{\widehat{\varkappa}}{\varkappa(t)} - 1\right) dt.$$

Let us turn now to Eq. $(2.8)_2$.

3.2.3 Effective thermal conductivity

Applying the averaging procedure to Eq. $(2.8)_2$, and taking into account the periodicity of the functions $\varkappa(\xi)$, $u_1(x,\xi)$, and $u_2(x,\xi)$), we obtain

$$\int\limits_0^1 \left(\frac{\partial}{\partial x}\left(\varkappa(\xi)\frac{\partial u_1(x,\xi)}{\partial \xi}\right) + \frac{\partial}{\partial x}\left(\varkappa(\xi)\frac{\partial u_0(x,\xi)}{\partial x}\right)\right) d\xi = f(x). \tag{2.16}$$

Since this equation is a consequence of Eq. $(2.8)_2$, it represents the necessary solvability condition for Eq. $(2.8)_2$ in the class of functions which are 1-periodic in ξ. We emphasize that the condition (2.16) is also sufficient for solvability of Eq. $(2.8)_2$ (see, [8], Ch. 2, Lemma 1).

Substituting the expressions (2.11) and (2.15) into Eq. (2.16), we find

$$\widehat{\varkappa}\frac{d^2 v_0}{dx^2} = f(x). \tag{2.17}$$

According (2.2), Eq. (2.17) should be subjected to the following boundary conditions:

$$v_0(0) = g_0, \quad v_0(1) = g_1.$$

Eq. (2.17) is called the *homogenized equation*, and the coefficient $\widehat{\varkappa}$ determined by formula (2.14) is called the *effective thermal conductivity.*

Finally, note that if $v_0(x)$ is the solution to Eq. (2.17), then Eq. $(2.8)_2$ can be solved for the function $u_2(x, \xi)$ in the class of functions which are 1-periodic in ξ. Hence, due to the relations (2.7) and (2.8), the expansion (2.5) formally satisfies Eq. (2.1) with the accuracy up to the terms of order ε.

3.2.4 Bakhvalov's method

Taking account of the expression (2.4), we rewrite Eq. (2.1) in the form

$$L_\varepsilon u = f(x), \tag{2.18}$$

where L_ε is a differential operator acting according to the formula

$$L_\varepsilon u = \frac{d}{dx}\left(\varkappa\left(\frac{x}{\varepsilon}\right)\frac{du}{dx}\right). \tag{2.19}$$

Following Bakhvalov [7], the solution to Eq. (2.18) will be sought in the form

$$u = v(x) + \sum_{i=1}^{\infty} \varepsilon^i N_i\left(\frac{x}{\varepsilon}\right)\frac{d^i v}{dx^i}. \tag{2.20}$$

Here $N_i(\xi)$ are 1-periodic functions, $v(x)$ is a unique solution to a certain homogenized problem (with constant coefficients), which will be derived below.

The function $v(x)$ does not depend on the fast variable ξ, and, moreover, it itself is represented in the form of asymptotic expansion

$$v(x) = \sum_{j=0}^{\infty} \varepsilon^j v_j(x). \tag{2.21}$$

Let us substitute the expansion (2.20) into the left-hand side of Eq. (2.18). Applying the chain rule of differentiation

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \varepsilon^{-1} \frac{\partial}{\partial \xi},$$

we obtain the following decomposition of the differential operator (2.19):

$$L_\varepsilon = \varepsilon^{-2} L_0 + \varepsilon^{-1} L_1 + L_2.$$

Here we introduced the notation

$$\begin{aligned} L_0 &= \frac{\partial}{\partial \xi}\Big(\varkappa(\xi)\frac{\partial}{\partial \xi}\Big), \\ L_1 &= \frac{\partial}{\partial \xi}\Big(\varkappa(\xi)\frac{\partial}{\partial x}\Big) + \frac{\partial}{\partial x}\Big(\varkappa(\xi)\frac{\partial}{\partial \xi}\Big), \quad L_2 = \varkappa(\xi)\frac{\partial^2}{\partial x^2}. \end{aligned}$$

We underline that in this notation, the variables x and ξ are regarded as independent, and the corresponding differential operations are assumed to be commutative.

Removing parentheses in the right-hand side of the identity

$$L_\varepsilon u = \big(\varepsilon^{-2} L_0 + \varepsilon^{-1} L_1 + L_2\big)\left(v + \sum_{i=1}^{\infty} \varepsilon^i N_i(\xi) \frac{d^i v}{dx^i}\right),$$

we obtain

$$\begin{aligned} L_\varepsilon u = {} & \varepsilon^{-2} L_0 v + \varepsilon^{-1}\Big(L_1 v + L_0 N_1 \frac{dv}{dx}\Big) \\ & + \varepsilon^0 \bigg(L_2 v + L_1 N_1 \frac{dv}{dx} + L_0 N_2 \frac{d^2 v}{dx^2}\bigg) \\ & + \sum_{i=1}^{\infty} \varepsilon^i \left(L_2 N_i \frac{d^i v}{dx^i} + L_1 N_{i+1} \frac{d^{i+1} v}{dx^{i+1}} + L_0 N_{i+2} \frac{d^{i+2} v}{dx^{i+2}}\right). \end{aligned}$$

Taking into account the fact that the function $v(x)$ does not depend on ξ, while L_1 and L_2 are differential operators of the first and second order with respect to the variable x, respectively, we arrive at the relation

$$\begin{aligned} L_\varepsilon u = {} & \varepsilon^{-1}\Big(\frac{d}{d\xi}\varkappa(\xi) + L_0 N_1\Big)\frac{dv}{dx} \\ & + \varepsilon^0\bigg(\varkappa(\xi) + \frac{d}{d\xi}\big(\varkappa(\xi)N_1\big) + \varkappa(\xi)\frac{dN_1}{d\xi} + L_0 N_2\bigg)\frac{d^2 v}{dx^2} \\ & + \sum_{i=1}^{\infty}\varepsilon^i\left(\varkappa(\xi)N_i + \frac{d}{d\xi}\big(\varkappa(\xi)N_{i+1}\big) + \varkappa(\xi)\frac{dN_{i+1}}{d\xi} + L_0 N_{i+2}\right)\frac{d^{i+2}v}{dx^{i+2}}. \end{aligned} \tag{2.22}$$

Since the equation $L_\varepsilon u = f(x)$ should be true, we try to choose the functions N_i in such a way that the term of order ε^{-1} will vanish and all terms of higher order will not depend on the fast variable ξ, i. e.,

$$L_0 N_1 + \frac{d}{d\xi}\varkappa(\xi) = 0, \tag{2.23}$$

$$L_0 N_i + \varkappa(\xi)\frac{dN_{i-1}}{d\xi} + \frac{d}{d\xi}\big(\varkappa(\xi)N_{i-1}\big) + \varkappa(\xi)N_{i-2} = h_i. \tag{2.24}$$

Here, h_i are constants $(i = 2, 3, \ldots)$, and, moreover, it is assumed that the normalization condition $N_0 = 1$ holds true.

Now, let us rewrite Eqs. (2.23), (2.24) in the form

$$L_0 N_i = F_i \quad (i = 1, 2, \ldots), \tag{2.25}$$

where the functions F_i, starting from the number $i = 2$, are expressed through the functions N_j with the index $j < i$. This circumstance can be used for the recurrence determining the functions N_i from Eqs. (2.24).

The constant (2.25) is determined from the condition of existence of 1-periodic solutions of Eq. (2.25). Applying the averaging operator $\langle\cdot\rangle$ to the both sides of Eq. (2.25), we obtain the equation $0 = \langle F_i\rangle$, from which it follows that

$$h_i = -\left\langle \varkappa(\xi)\frac{dN_{i-1}}{d\xi} + \frac{d}{d\xi}\big(\varkappa(\xi)N_{i-1}\big) + \varkappa(\xi)N_{i-2}\right\rangle. \tag{2.26}$$

It is not hard to verify that the existence condition in the class of 1-periodic functions for Eq. (2.23), which can be written as

$$\frac{d}{d\xi}\bigg(\varkappa(\xi)\Big(\frac{dN_1}{d\xi} + 1\Big)\bigg) = 0, \tag{2.27}$$

holds true owing to the periodicity of the function $\varkappa(\xi)$.

For $i = 1$, we have

$$h_1 = -\left\langle \frac{d}{d\xi}\varkappa(\xi) \right\rangle = \varkappa(1) - \varkappa(0) = 0.$$

From here it follows that the periodic solution to Eq. (2.25), $i = 1$, exists when $h_1 = 0$.

Observe that the solution $N_i(\xi)$ to Eq. (2.25) is determined up to an additive constant. Therefore, to fix the solution we put

$$N_i(0) = 0. \tag{2.28}$$

Then, the solution to the problem (2.25), (2.28) under the condition (2.26) exists and is unique.

The problem (2.25), (2.28) is called the *problem on the periodic cell.*

Thus, since the functions N_i satisfy the relations (2.23), (2.24), the expansion (2.22) takes the form

$$L_\varepsilon u = \sum_{i=2}^{\infty} \varepsilon^{i-2} h_i \frac{d^i v}{dx^i}.$$

Hence, the function v should satisfy the equation

$$\sum_{i=2}^{\infty} \varepsilon^{i-2} h_i \frac{d^i v}{dx^i} = f(x). \tag{2.29}$$

Let us turn to the boundary conditions (2.2). Substituting the series (2.20) into the relations (2.2), we obtain

$$v(0) + \sum_{i=1}^{\infty} \varepsilon^i N_i(0) \frac{d^i v}{dx^i}(0) = g_0, \tag{2.30}$$

$$v(1) + \sum_{i=1}^{\infty} \varepsilon^i N_i\left(\frac{1}{\varepsilon}\right) \frac{d^i v}{dx^i}(1) = g_1. \tag{2.31}$$

By virtue of the choice of the rational value of parameter ε (see formula (2.3)) and the periodicity of the function $N_i(\xi)$, the following relations take place:

$N_i(n) = N_i(0)$. Taking into account the condition (2.28), we derive from the relations (2.30) and (2.31) the following boundary conditions:

$$v(0) = g_0, \quad v(1) = g_1. \tag{2.32}$$

Thus, if the function $v(x)$ formally satisfies the relations (2.29), (2.32), then the series (2.20) is the formal asymptotic expansion for the original problem (2.1), (2.2).

Eq. (2.29) is called the *formal homogenized equation* of the infinite order of accuracy. The coefficient h_2 of the main part of Eq. (2.29) is called the *average thermal conductivity coefficient* of the layered medium and is denoted by $\widehat{\varkappa}$.

Let us calculate the average thermal conductivity coefficient $\widehat{\varkappa}$. First, according to Eq. (2.26), we have

$$\widehat{\varkappa} = h_2 = -\left\langle \varkappa(\xi)\Big(\frac{dN_1}{d\xi} + 1\Big)\right\rangle,$$

where the function N_1 is the solution to Eq. (2.27).

Integrating Eq. (2.27), we obtain

$$\varkappa(\xi)\Big(\frac{dN_1}{d\xi} + 1\Big) = C_1,$$

where C_1 is a constant. Thus, we get

$$\frac{dN_1}{d\xi} = \frac{C_1}{\varkappa(\xi)} - 1. \tag{2.33}$$

Now, we apply the averaging operator $\langle\cdot\rangle$ to the both sides of Eq. (2.33). Taking into account the 1-periodicity of the function N_1, we obtain

$$0 = \big\langle \varkappa(\xi)^{-1}\big\rangle\, C_1 - 1.$$

From here we determine $C_1 = \langle \varkappa(\xi)^{-1}\rangle^{-1}$ and, consequantly,

$$\widehat{\varkappa} = \big\langle \varkappa(\xi)^{-1}\big\rangle^{-1}. \tag{2.34}$$

Further, the homogenized equation (2.29) contains the small parameter ε and admits an asymptotic solution in the form of expansion (2.21). At that, the substitution of the series (2.21) into the right-hand side of Eq. (2.29) yields

$$\sum_{i=2}^{\infty} \varepsilon^{i-2} h_i \sum_{j=0}^{\infty} \varepsilon^j \frac{d^i v_j}{dx^i} = \sum_{i=2}^{\infty} \sum_{j=0}^{\infty} \varepsilon^{i+j-2} h_i \frac{d^i v_j}{dx^i}.$$

Changing the indexes by the formulas $q = i + j - 2$ and $j = j$, we arrive at the equation

$$\sum_{q=0}^{\infty} \varepsilon^q \left(h_2 \frac{d^2 v_q}{dx^2} + \sum_{j=0}^{q-1} h_{q-j+2} \frac{d^{q-j+2} v_j}{dx^{q-j+2}} \right) = f(x).$$

Correspondingly, substituting the series (2.21) into the boundary conditions (2.32), we obtain

$$\sum_{q=0}^{\infty} \varepsilon^q v_q(0) = g_0, \quad \sum_{q=0}^{\infty} \varepsilon^q v_q(1) = g_1.$$

Thus, the function $v_0(x)$, which is the leading term of the asymptotic expansion (2.21), should satisfy the equation

$$h_2 \frac{d^2 v_0}{dx^2} = f(x) \tag{2.35}$$

and the boundary conditions

$$v_0(0) = g_0, \quad v_0(1) = g_1. \tag{2.36}$$

Further, the other terms of the series (2.21) should satisfy the equation

$$h_2 \frac{d^2 v_q}{dx^2} = -\sum_{j=0}^{q-1} h_{q-j+2} \frac{d^{q-j+2} v_j}{dx^{q-j+2}} \qquad (q = 1, 2, \dots) \tag{2.37}$$

and the boundary conditions

$$v_q(0) = 0, \quad v_q(1) = 0. \tag{2.38}$$

Thus, all the functions $v_q(x)$ can be determined step by step from the recurrence sequence of the boundary value problems (2.37), (2.38), to which we have reduced the procedure of constructing the formal asymptotic solution of the problem (2.1), (2.2).

The problem (2.35), (2.36) is called the *homogenized problem of the zeroth order.* It describes the temperature field in a such homogeneous medium whose properties in a sense are close to the effective properties of the original non-homogeneous medium.

3.3 Homogenization of the heat-conduction process in composite materials

3.3.1 Problem statement

Let Q be an n-dimensional ($n = 2, 3$) cube $\big\{\xi \in \mathbb{R}^n : |\xi_k| < 1/2,\ k = 1, \dots, n\big\}$. We assume that a domain ω is embedded into Q, and this structure is periodically extended onto the entire space $\mathbb{R}^n$ with the period 1. We denote by $\mathcal{B}$ the union of all the obtained domains corresponding to the inclusions ω. We will assume that the matrix domain $\mathcal{M} = \mathbb{R}^n \setminus \overline{\mathcal{B}}$ is connected.

In the case of anisotropic components of the composite material, its heat-conduction properties are described by a symmetric positive-definite matrix $\|\varkappa_{jk}(\xi)\|$, where

$$\varkappa_{jk}(\xi) = \begin{cases} \varkappa^{\mathrm{i}}_{jk}, & \xi \in \mathcal{B}, \\ \varkappa^{\mathrm{m}}_{jk}, & \xi \in \mathcal{M}. \end{cases}$$

The functions $\varkappa_{jk}(\xi)$ are periodic with respect to each variable ξ_k ($k = 1, \dots, n$) with the period 1. The matrix $\|\varkappa_{jk}(\xi)\|$ determines the heat-conduction tensor.

Let Ω be a bounded domain in $\mathbb{R}^n$ with the boundary Γ. The temperature field $u(x)$ satisfies the differential equation

$$L_\varepsilon u(x) \equiv \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(\varkappa_{jk}(\varepsilon^{-1}x)\frac{\partial u}{\partial x_k}\right) = f(x), \quad x \in \Omega \setminus \Sigma_\varepsilon, \qquad (3.1)$$

in the entire domain Ω outside the interface surface Σ_ε between the inclusions and the matrix. Across the surfaces of discontinuity of the conductivity coefficients, we impose the following continuity conditions for the temperature and the heat flux density:

$$[u]\Big|_{\Sigma_\varepsilon} = 0, \quad \left[\frac{\partial u}{\partial \nu}\right]\Bigg|_{\Sigma_\varepsilon} = 0. \qquad (3.2)$$

Here, $\partial/\partial\nu$ is the conormal derivative (n_j is the cosine of the angle between the normal and the j-th coordinate direction)

$$\frac{\partial u}{\partial \nu} = \sum_{j,k=1}^{n} \varkappa_{jk}(\varepsilon^{-1}x)\frac{\partial u}{\partial x_k}\, n_j. \qquad (3.3)$$

To fix the idea we assume that the boundary Γ of the domain Ω is maintained under a constant temperature, i. e.,

$$u(x) = 0, \quad x \in \Gamma. \tag{3.4}$$

Following Bakhvalov and Panasenko [8], we will construct the asymptotics of the solution to the problem (3.1) – (3.4) under the smoothness assumption on the function $f(x)$ in the closed domain $\overline{\Omega}$.

3.3.2 Asymptotics of the solution

Applying the multiple scale method, we will construct an approximate solution to the problem (3.1) – (3.4) in the form

$$u = u_0(x,\xi) + \varepsilon u_1(x,\xi) + \varepsilon^2 u_2(x,\xi) + \ldots, \tag{3.5}$$

where the functions $u_i(x,\xi)$ are periodic with respect to each coordinate ξ_k ($k = 1, \ldots, n$) with the period 1. Observe that in formula (3.5), along with the slow variable x, we introduced the fast variable

$$\xi = \frac{x}{\varepsilon}. \tag{3.6}$$

Substituting the expansion (3.5) into Eq. (3.1), we make use of the chain rule of differentiation

$$\frac{d}{dx_k} F\Big(x, \frac{x}{\varepsilon}\Big) = \bigg(\frac{\partial F(x,\xi)}{\partial x_k} + \varepsilon^{-1}\frac{\partial F(x,\xi)}{\partial \xi_k}\bigg)\bigg|_{\xi=\varepsilon^{-1}x}. \tag{3.7}$$

The following decomposition for the differential operator appearing in Eq. (3.1) holds true:

$$L_\varepsilon u(x,\xi) = \big(\varepsilon^{-2}L_0 + \varepsilon^{-1}L_1 + L_2\big)u(x,\xi).$$

Here we introduced the notation

$$L_0 = L_{\xi\xi}, \quad L_1 = L_{x\xi} + L_{\xi x}, \quad L_2 = L_{xx};$$

$$L_{\alpha\beta}u(x,\xi) = \sum_{j,k=1}^{n} \frac{\partial}{\partial \alpha_j}\bigg(\varkappa_{jk}(\xi)\frac{\partial u(x,\xi)}{\partial \beta_k}\bigg).$$

As a results of the substitution of the expansion (3.5) into Eq. (3.1), we obtain

$$0 = \big[\varepsilon^{-2} L_0 u_0 + \varepsilon^{-1}(L_1 u_0 + L_0 u_1) \\ + \varepsilon^0 (L_2 u_0 + L_1 u_1 + L_0 u_2 - f) + \ldots\big]\Big|_{\xi=\varepsilon^{-1}x}. \tag{3.8}$$

Now let us require that the terms of order ε^{-2}, ε^{-1}, and ε^0 in the expansion above are equal to zero. At that, the term of order ε leaves a discrepancy, and, therefore, the expansion (3.5) will satisfy Eq. (3.1) with the accuracy up to terms of order ε.

So, for the recurrence determination of the functions $u_0(x,\xi)$, $u_1(x,\xi)$, and $u_2(x,\xi)$, we obtain the following system of differential equations with respect to the variable ξ containing x as a parameter:

$$L_0 u_0 = 0, \tag{3.9}$$

$$L_0 u_1 = -L_1 u_0, \tag{3.10}$$

$$L_0 u_2 = -L_1 u_1 - L_2 u_0 + f. \tag{3.11}$$

Further, let us substitute the expansion (3.5) into the left-hand sides of the contact conditions (3.2) imposed on the interface Σ_ε. We have

$$[u]\Big|_{\Sigma_\varepsilon} = \Big(\big[u_0(x,\xi)\big]\Big|_{\xi\in\Sigma} + \varepsilon\big[u_1(x,\xi)\big]\Big|_{\xi\in\Sigma} \\ + \varepsilon^2\big[u_2(x,\xi)\big]\Big|_{\xi\in\Sigma} + \ldots\Big)\Big|_{x=\varepsilon\xi}. \tag{3.12}$$

Here, $\Sigma = \partial\mathcal{B}$ is the interface surface between the inclusions and the matrix in the scratched coordinates. Introducing the notation

$$\frac{\partial u(x,\xi)}{\partial \nu_\alpha} = \sum_{j,k=1}^{n} \varkappa_{jk}(\xi) \frac{\partial u(x,\xi)}{\partial \alpha_k}\, n_j,$$

we can write

$$\left[\frac{\partial u}{\partial \nu}\right]\Bigg|_{\Sigma_\varepsilon} = \left(\varepsilon^{-1}\left[\frac{\partial u_0}{\partial \nu_\xi}\right]\Bigg|_{\xi\in\Sigma} + \left[\frac{\partial u_1}{\partial \nu_\xi} + \frac{\partial u_0}{\partial \nu_x}\right]\Bigg|_{\xi\in\Sigma} \\ + \varepsilon\left[\frac{\partial u_2}{\partial \nu_\xi} + \frac{\partial u_1}{\partial \nu_x}\right]\Bigg|_{\xi\in\Sigma} + \ldots\right)\Bigg|_{x=\varepsilon\xi}. \tag{3.13}$$

Considering now the variable x in the relations (3.12), (3.13) as a parameter, we will require that according to the homogeneous conditions (3.2) the three main terms in both conditions (3.12), (3.13) should be equal to zero. Thus, in addition to Eqs. (3.9) – (3.11) we obtain

$$\big[u_0(x,\xi)\big]\Big|_{\xi\in\Sigma} = 0, \quad \left[\frac{\partial u_0}{\partial \nu_\xi}\right]\Bigg|_{\xi\in\Sigma} = 0; \tag{3.14}$$

$$\big[u_1(x,\xi)\big]\Big|_{\xi\in\Sigma} = 0, \quad \left[\frac{\partial u_1}{\partial \nu_\xi} + \frac{\partial u_0}{\partial \nu_x}\right]\Bigg|_{\xi\in\Sigma} = 0; \tag{3.15}$$

$$\big[u_2(x,\xi)\big]\Big|_{\xi\in\Sigma} = 0, \quad \left[\frac{\partial u_2}{\partial \nu_\xi} + \frac{\partial u_1}{\partial \nu_x}\right]\Bigg|_{\xi\in\Sigma} = 0. \tag{3.16}$$

The solvability of the problems for determining the functions $u_i(x,\xi)$ can be established with help of the following statement (see, [8], Ch. 4, § 1, Lemma 2):

L e m m a. *Let* $\varkappa_{jk}(\xi)$, $F_0(\xi)$, $F_k(\xi)$ *be* 1*-periodic piecewise-smooth functions, and the matrix* $\|\varkappa_{jk}(\xi)\|$ *is symmetric and positive-definite at every point* ξ, *i. e.,*

$$\varkappa_{jk}(\xi) = \varkappa_{kj}(\xi); \quad \varkappa_{jk}(\xi)\eta_j\eta_k \geq \varkappa_1\eta_j\eta_j \ \forall \eta \in \mathbb{R}^n, \tag{3.17}$$

where $\varkappa_1 > 0$ *is a constant which does not depend on* ξ.

Then, for the existence of 1*-periodic solution to the problem*

$$L_{\xi\xi}N = F_0(\xi) + \frac{\partial}{\partial \xi_k}F_k(\xi), \quad \xi \notin \Sigma; \tag{3.18}$$

$$\big[N\big]\Big|_{\xi\in\Sigma} = 0, \tag{3.19}$$

$$\left[\left(\varkappa_{jk}(\xi)\frac{\partial N}{\partial \xi_j} - F_k(\xi)\right)n_k\right]\Bigg|_{\xi\in\Sigma} = 0 \tag{3.20}$$

it is necessary and sufficient that $\langle F_0\rangle = 0$. Here the summation over a repeated index $1 \leq k \leq n$ is assumed, and the following notation is introduced for the average over period

$$\langle F(x,\xi)\rangle = \int\limits_{-1/2}^{1/2} \dots \int\limits_{-1/2}^{1/2} F(x,\xi_1,\dots,\xi_n)\, d\xi_1,\dots d\xi_n.$$

Recall that in the multiple integral above the variables x and ξ are considered as independent variables.

The general 1-periodic solution to the problem (3.18) – (3.20) can be written in the form $N(\xi) = \overline{N}(\xi) + C$, where $\overline{N}(\xi)$ is a unique solution with the zero average over period, i. e., $\langle \overline{N} \rangle = 0$, and C is an arbitrary constant.

3.3.3 Homogenized problem

It is clear that all the conditions of the Lemma are satisfied for the problem (3.9), (3.14), and consequently, its solution will be a function which does not depend on the fast variable ξ, i. e.,

$$u_0(x, \xi) = v_0(x). \tag{3.21}$$

Then, Eq. (3.10) and the contact condition (3.15) can be rewritten as follows:

$$L_{\xi\xi} u_1 = -\frac{\partial}{\partial \xi_k}\left(\varkappa_{jk}(\xi)\frac{\partial v_0}{\partial x_j}\right), \tag{3.22}$$

$$\left[u_1\right]\Big|_{\xi\in\Sigma} = 0, \quad \left[\left(\varkappa_{jk}(\xi)\frac{\partial u_1}{\partial \xi_j} + \varkappa_{jk}(\xi)\frac{\partial v_0}{\partial x_j}\right)n_k\right]\Bigg|_{\xi\in\Sigma} = 0. \tag{3.23}$$

From the Lemma it follows that the problem (3.22), (3.23) has a 1-periodic solution which can be represented in the form

$$u_1(x, \xi) = N_i(\xi)\frac{\partial v_0}{\partial x_i}, \tag{3.24}$$

where $N_i(\xi)$ $(i = 1, \dots, n)$ is a 1-periodic solution to the problem

$$L_{\xi\xi}(N_i + \xi_i) = 0, \quad \xi \notin \Sigma; \tag{3.25}$$

$$\left[N_i\right]\Big|_{\xi\in\Sigma} = 0, \quad \left[\frac{\partial(N_i + \xi_i)}{\partial \nu_\xi}\right]\Bigg|_{\xi\in\Sigma} = 0. \tag{3.26}$$

The problem (3.25), (3.26) is called the *problem on the periodic cell.* By virtue of the Lemma, its solution is determined up to an arbitrary constant, which can be fixed by the condition $\langle N_i \rangle = 0$.

Further, taking into account the expressions (3.21) and (3.24), we rewrite Eq. (3.11) as follows:

$$L_{\xi\xi}u_2 + \frac{\partial}{\partial \xi_k}\big(\varkappa_{jk}(\xi)N_i(\xi)\big)\frac{\partial^2 v_0}{\partial x_j \partial x_i} + \left(\varkappa_{jk}(\xi)\frac{\partial N_i(\xi)}{\partial x_j}\right)\frac{\partial^2 v_0}{\partial x_k \partial x_i} + \varkappa_{jk}(\xi)\frac{\partial^2 v_0}{\partial x_k \partial x_j} = f(x).$$

From here, changing the notation of indexes, we derive

$$L_{\xi\xi}u_2 + \left(\frac{\partial}{\partial \xi_k}\big(\varkappa_{ki}(\xi)N_l(\xi)\big) + \varkappa_{ij}(\xi)\frac{\partial N_l(\xi)}{\partial x_j} + \varkappa_{il}(\xi)\right)\frac{\partial^2 v_0}{\partial x_i \partial x_l} = f(x). \quad (3.27)$$

Correspondingly, the second contact condition (3.16) takes the form

$$\left[\left(\varkappa_{kj}(\xi)\frac{\partial u_2}{\partial \xi_j} + \varkappa_{ki}(\xi)N_l(\xi)\frac{\partial^2 v_0}{\partial x_i \partial x_l}\right)n_k\right]\Bigg|_{\xi\in\Sigma} = 0. \quad (3.28)$$

The necessary and sufficient condition for the solvability of the boundary value problem (3.27), (3.28) in the class of 1-periodic functions yields the homogenized equation

$$\widehat{\varkappa}_{il}\frac{\partial^2 v_0}{\partial x_i \partial x_l} = f(x), \quad (3.29)$$

where

$$\widehat{\varkappa}_{il} = \left\langle \varkappa_{ij}(\xi)\frac{\partial N_l(\xi)}{\partial \xi_j} + \varkappa_{il}(\xi)\right\rangle. \quad (3.30)$$

Finally, substituting the expansion (3.5) in the boundary condition (3.4), we find that Eq. (3.29) should be subjected to the boundary condition

$$v_0(x) = 0, \quad x \in \Gamma. \quad (3.31)$$

Thus, with the accuracy up to terms of order ε, the solution to the problem (3.1) – (3.4) coincides with a smooth function $v_0(x)$, which does not depend on the parameter ε. This function is the unique solution to the homogenized problem (3.29), (3.31) with constant coefficients (3.30).

The matrix of effective conductivity coefficients $\|\widehat{\varkappa}_{il}\|$ is symmetric and positive-definite (see, in particular, [8], Ch. 4, § 1, Theorem 1) and determines the conductivity tensor of a such hyposetical homogeneous material whose properties are close to the average properties of the original composite material.

The question of determining the coefficients $\widehat{\varkappa}_{il}$ is related to constructing a numerical solution N_i $(i = 1, \dots, n)$ to the problem on the periodic cell (3.25), (3.26) together with the subsequent calculations based on formula (3.30).

3.3.4 Homogenization of the heat-conduction process in a periodic porous medium

Let Q be an n-dimensional $(n = 2, 3)$ cube $\{\xi \in \mathbb{R}^n : |\xi_k| < 1/2, \ k = 1, \dots, n\}$. Taking away some portion ω from the cube Q, we obtain the domain $\overline{Q} \setminus \overline{\omega}$, which will be periodically extended onto the entire space $\mathbb{R}^n$ with the period 1. We denote by $\mathcal{M}$ the union of all the obtained domains. The domain $\mathcal{M}$ is assumed to be connected. Furthermore, we denote by ε a small positive parameter and introduce a set $\mathcal{M}_\varepsilon$ obtained from the set $\mathcal{M}$ by the homothetic transformation with the extension coefficient ε^{-1}, that is

$$\mathcal{M}_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon^{-1} x \in \mathcal{M}\}.$$

Let also Ω be a bounded domain in $\mathbb{R}^n$ with the boundary Γ. Then, $\Omega_\varepsilon = \Omega \cap \mathcal{M}_\varepsilon$ is a perforated domain with the boundary $\partial\Omega_\varepsilon$, which is composed of the following two parts: $\partial\mathcal{M}_\varepsilon \cap \Omega$ and $\Gamma \cap \mathcal{M}_\varepsilon$. The temperature field $u(x)$ in the periodically perforated anisotropic porous medium Ω_ε satisfies the differential equation

$$L_\varepsilon u(x) \equiv \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \left(\varkappa_{jk}(\varepsilon^{-1} x) \frac{\partial u}{\partial x_k} \right) = f(x), \quad x \in \Omega_\varepsilon. \tag{3.32}$$

On the boundaries of the cavities $\partial\mathcal{M}_\varepsilon \cap \Omega$, we impose the homogeneous boundary condition of the second type

$$\left. \frac{\partial u}{\partial \nu} \right|_{\partial\mathcal{M}_\varepsilon \cap \Omega} = 0, \tag{3.33}$$

where $\partial u / \partial \nu$ is the conormal derivative (3.3). This boundary condition means that there is no heat exchange between the surface and the medium inside of the cavities.

On the external surface $\Gamma \cap \mathcal{M}_\varepsilon$, we assume that a zero temperature is maintained, i. e.,

$$u\Big|_{\Gamma\cap\mathcal{M}_\varepsilon} = 0. \tag{3.34}$$

Following Bakhvalov and Panasenko [8], the asymptotic solution to the problem (3.32) – (3.34) under the smoothness assumption on the function $f(x)$, $x \in \overline{\Omega}$, will be constructed in the form

$$u = u_0(x,\xi) + \varepsilon u_1(x,\xi) + \varepsilon^2 u_2(x,\xi) + \ldots, \tag{3.35}$$

where $\xi = \varepsilon^{-1}x$ are the fast coordinates (3.6), $u_i(x,\xi)$ are smooth functions which are periodic with respect to each variable ξ_k ($k = 1, \ldots, n$) with the period 1.

Let us substitute the expansion (3.35) into Eq, (3.32), making use of the chain rule of differentiation (3.7). After collecting terms of equal order in ε, we arrive at the relation (3.8), which yields Eqs. (3.9) – (3.11).

Now, substituting the expansion (3.35) into the left-hand side of the boundary condition (3.33), we obtain

$$\frac{\partial u}{\partial \nu}\bigg|_{\partial\mathcal{M}_\varepsilon\cap\Omega} = \Bigg(\varepsilon^{-1}\frac{\partial u_0}{\partial \nu_\xi}\bigg|_{\xi\in\partial\mathcal{M}} + \bigg(\frac{\partial u_1}{\partial \nu_\xi} + \frac{\partial u_0}{\partial \nu_x}\bigg)\bigg|_{\xi\in\partial\mathcal{M}} + \\ + \varepsilon\bigg(\frac{\partial u_2}{\partial \nu_\xi} + \frac{\partial u_1}{\partial \nu_x}\bigg)\bigg|_{\xi\in\partial\mathcal{M}} + \ldots\Bigg)\bigg|_{\xi=\varepsilon^{-1}x}.$$

In view of the homogeneous boundary condition (3.33), we require that the following relation should be satisfied:

$$\frac{\partial u_0}{\partial \nu_\xi}\bigg|_{\xi\in\partial\mathcal{M}} = 0, \tag{3.36}$$

$$\bigg(\frac{\partial u_1}{\partial \nu_\xi} + \frac{\partial u_0}{\partial \nu_x}\bigg)\bigg|_{\xi\in\partial\mathcal{M}} = 0, \tag{3.37}$$

$$\bigg(\frac{\partial u_2}{\partial \nu_\xi} + \frac{\partial u_1}{\partial \nu_x}\bigg)\bigg|_{\xi\in\partial\mathcal{M}} = 0. \tag{3.38}$$

The solvability of the problems for the functions $u_i(x,\xi)$ can be established with the help of the following statement (see [8], Ch. 4, § 3, Lemma 1):

L e m m a. *Let* $\varkappa_{jk}(\xi)$, $F_0(\xi)$, $F_k(\xi)$ *be* 1*-periodic functions. We assume that the functions* $\varkappa_{jk}(\xi)$ *satisfy the conditions* (3.17).

Then the necessary and sufficient condition for existence of a 1*-periodic solution to the equation*

$$L_{\xi\xi}N = F_0(\xi) + \frac{\partial F_k(\xi)}{\partial \xi_k}, \quad \xi \in \mathcal{M}$$

subjected to the boundary condition

$$\left(\varkappa_{jk}(\xi)\frac{\partial}{\partial \xi_j}N(\xi) - F_k(\xi)\right)n_k\Big|_{\xi\in\partial\mathcal{M}} = 0,$$

is as follows:

$$\langle F_0(\xi)\rangle^{\mathcal{M}} \equiv \int\limits_{Q\cap\mathcal{M}} F_0(\xi)\,d\xi = 0. \tag{3.39}$$

From this Lemma, it follows that the first problems (3.9), (3.36) and (3.10), (3.37) are solvable. Moreover, by analogy with formulas (3.21) and (3.24), the following representations hold true:

$$u_0(x,\xi) = v_0(x), \quad u_1(x,\xi) = N_i(\xi)\frac{\partial v_0}{\partial x_i}. \tag{3.40}$$

Here, $N_i(\xi)$ are the normalized (with the help of the condition $\langle N_i\rangle^{\mathcal{M}} = 0$) 1-periodic solutions to the problems on the periodic cell

$$L_{\xi\xi}(N_i + \xi_i) = 0, \quad \xi \in \mathcal{M}; \tag{3.41}$$

$$\frac{\partial}{\partial \nu_\xi}(N_i + \xi_i)\Big|_{\xi\in\partial\mathcal{M}} = 0. \tag{3.42}$$

Taking into account the expressions (3.40), we rewrite Eq. (3.11) in the form (3.27), while the boundary condition (3.38) is represented as follows:

$$\left(\varkappa_{jk}(\xi)\frac{\partial u_2(x,\xi)}{\partial \xi_j} + \varkappa_{jk}(\xi)N_l(\xi)\frac{\partial^2 v_0}{\partial x_i \partial x_l}\right)n_k\Big|_{\xi\in\partial\mathcal{M}} = 0. \tag{3.43}$$

The necessary and sufficient condition for solvability of the problem (3.27), (3.43) in the class of 1-periodic functions (see, formula (3.39)) is equivalent to the relation

$$\left\langle \left(\varkappa_{ij}(\xi)\frac{\partial N_l}{\partial \xi_j} + \varkappa_{il}(\xi)\right)\frac{\partial^2 v_0}{\partial x_i \partial x_l}\right\rangle^{\mathcal{M}} = \langle f(x)\rangle^{\mathcal{M}},$$

or, that is the same,

$$\widehat{\varkappa}_{il}\frac{\partial^2 v_0}{\partial x_i \partial x_l} = f(x). \tag{3.44}$$

Here we introduced the notation

$$\widehat{\varkappa}_{il} = |Q \setminus \omega|^{-1} \left\langle \varkappa_{ij}(\xi)\frac{\partial N_l(\xi)}{\partial \xi_j} + \varkappa_{il}(\xi) \right\rangle^{\mathcal{M}}, \tag{3.45}$$

where $|Q \setminus \omega|$ is the volume (area for $n = 2$) of the domain $Q \setminus \omega$ such that

$$|Q \setminus \omega| = \int\limits_{Q\setminus\omega} d\xi.$$

Finally, substituting the expansion (3.35) into the boundary condition (3.34), we find that the homogenized equation (3.44) should be subjected to the following boundary condition:

$$v_0\Big|_{\Gamma} = 0. \tag{3.46}$$

Thus, with the accuracy up to terms of order ε the solution of the problem (3.32) – (3.34) coincides with a smooth function $v_0(x)$, which does not depend on the parameter ε. This function is the unique solution of the homogenized problem (3.44), (3.46) with the constant coefficients (3.45).

3.3.5 Symmetry of the effective heat conduction coefficients

Let us introduce the notation $M_i(\xi) = N_i(\xi) + \xi_i$ $(i = 1, \ldots, n)$. According to the relations (3.41), (3.42) the function $M_i(\xi)$ satisfies the problem

$$L_{\xi\xi}M_i = 0, \quad \xi \in \mathcal{M}; \qquad \frac{\partial M_i}{\partial \nu_\xi}\Big|_{\xi\in\partial\mathcal{M}} = 0. \tag{3.47}$$

Multiplying both sides of Eq. $(3.47)_1$ by $\varphi(\xi)$ and integrating over the domain $Q \setminus \omega$, we obtain

$$\left\langle \frac{\partial}{\partial \xi_k}\Big(\varkappa_{kj}(\xi)\frac{\partial M_l}{\partial \xi_j}\Big)\varphi(\xi) \right\rangle^{\mathcal{M}} = 0.$$

Now, for any 1-periodic differentiable function $\varphi(\xi)$, using the integration by parts with the homogeneous boundary condition $(3.47)_2$ taken into account, we establish the following identity:

$$\left\langle \varkappa_{kj}(\xi)\frac{\partial M_l}{\partial \xi_j}\frac{\partial \varphi}{\partial \xi_k}\right\rangle^{\mathcal{M}} = 0 \quad (l = 1, \ldots, n). \tag{3.48}$$

Let us rewrite formula (3.45) for the effective heat conduction coefficients as follows (see the proof of Theorem 1 in [8], Ch. 4, § 1):

$$|Q \setminus \omega|\widehat{\varkappa}_{il} = \left\langle \varkappa_{ij}(\xi)\frac{\partial M_l}{\partial \xi_j}\right\rangle^{\mathcal{M}} = \left\langle \varkappa_{kj}(\xi)\frac{\partial M_l}{\partial \xi_j}\frac{\partial \xi_i}{\partial \xi_k}\right\rangle^{\mathcal{M}}.$$

Adding the left-hand side of formula (3.48) with $\varphi(\xi) = N_i(\xi)$ to the right-hand side of the relation obtained above, we get

$$|Q \setminus \omega|\, \widehat{\varkappa}_{il} = \left\langle \varkappa_{kj}(\xi)\frac{\partial M_l}{\partial \xi_j}\frac{\partial M_i}{\partial \xi_k}\right\rangle^{\mathcal{M}}. \tag{3.49}$$

Formula (3.49) and the symmetry of the coefficients $\varkappa_{jk}$ imply the symmetry of the coefficients $\widehat{\varkappa}_{il}$.

Note that from the assumption that the matrix $\|\varkappa_{jk}\|$ is positive definite it follows that the matrix of effective heat conduction coefficients $\|\widehat{\varkappa}_{il}\|$ is also positive definite (see, [8], Ch. 4).

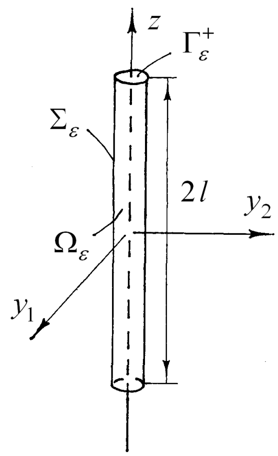

Figure 4.1: The rod geometry and the coordinate system.

3.4 Asymptotic modeling of the heat conduction in a thin rod

3.4.1 Heat-conduction in a thin rod

We consider the problem of stationary heat-conduction in a thin cylindrical rod Ω_ε with the constant cross-section ω_ε. Here, ε is a small positive parameter, which is equal to the ratio of the diameter of the cross-section ω_ε to the rod's length $2l$. Correspondingly, the thin rod Ω_ε is obtained from the cylinder $\omega \times (-l, l)$ by compressing in the transverse directions by ε^{-1} times, i. e.,

$$\Omega_\varepsilon = \big\{x = (y, z) : \ \varepsilon^{-1} y \in \omega, \ z \in (-l, l)\big\}.$$

It is assumed that ω is a fixed plane domain bounded by a simple smooth closed contour Γ (see Fig. 4.1).

The temperature distribution $u(x)$ in the inner points of the rod is described by the Poisson equation

$$\frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2} + \frac{\partial^2 u}{\partial z^2} + \varkappa^{-1} f(x) = 0, \tag{4.1}$$

where $f(x)$ is the density of volume heat sources, $\varkappa$ is the heat conductivity.

Let us assume that the rod lateral surface

$$\Sigma_\varepsilon = \big\{x : \ \varepsilon^{-1} y \in \Gamma, \ z \in (-l, l)\big\}$$

exchanges heat with the surrounding medium according to Newton's law

$$\frac{\partial u}{\partial n} + ku\Big|_{\Sigma_\varepsilon} = 0. \tag{4.2}$$

To fix our ideas, we assume that

$$u\Big|_{\Gamma_\varepsilon^\pm} = u_0^\pm(y), \tag{4.3}$$

that is the rod ends $\Gamma_\varepsilon^\pm = \{x : \ \varepsilon^{-1}y \in \omega, \ z = \pm l\}$ are maintained at a prescribed temperature.

3.4.2 Quasi one-dimensional process of heat-conduction in a thin rod

Following Zino and Tropp [23], we will construct an approximate (as $\varepsilon \to 0$) solution to the problem (4.1) – (4.3) under the smallness assumption on the heat exchange coefficient

$$k = \varepsilon k_0. \tag{4.4}$$

Let us introduce the following stretched variables in Eq. (4.1) and the boundary condition (4.2):

$$\eta = (\eta_1, \eta_2), \quad \eta = \varepsilon^{-1}y. \tag{4.5}$$

Taking into account the relation (4.4), we will have

$$\varepsilon^{-2}\left(\frac{\partial^2 u}{\partial \eta_1^2} + \frac{\partial^2 u}{\partial \eta_2^2}\right) + \frac{\partial^2 u}{\partial z^2} + \varkappa^{-1} f(\varepsilon\eta, z) = 0, \tag{4.6}$$

$$\varepsilon^{-1}\frac{\partial u}{\partial \nu} + \varepsilon k_0 u = 0, \tag{4.7}$$

where $\partial/\partial\nu$ is the normal derivative with respect to the outward normal to the contour $\partial\omega = \Gamma$ on the plane of coordinates η_1, η_2.

Far from the rod ends, the solution to the problem (4.6), (4.7), (4.3) will be constructed in the form of (*outer*) asymptotic expansion

$$u(x) = U_0(\eta, z) + \varepsilon^2 U_2(\eta, z) + \dots . \tag{4.8}$$

After substituting the expansion (4.8) into the relations (4.6), (4.7) and collecting terms, we equate to zero the coefficients at subsequent powers of ε. As a result, we derive the following relations:

$$\Delta_\eta U_0(\eta, z) = 0,\ \ \eta \in \omega; \quad \frac{\partial U_0}{\partial \nu}(\eta, z) = 0,\ \ \eta \in \Gamma; \tag{4.9}$$

$$-\Delta_\eta U_2 = \frac{\partial^2 U_0}{\partial z^2} + \varkappa^{-1} f,\ \ \eta \in \omega; \quad \frac{\partial U_2}{\partial \nu}\Big|_\Gamma = -k_0 U_0\Big|_\Gamma. \tag{4.10}$$

The solution of the homogeneous Neumann problem (4.9) is an arbitrary function $U_0(z)$ of the longitudinal coordinate z (to be determined in the sequel). The problem (4.10) as the two-dimensional Neumann problem for the domain ω bounded by the contour Γ has a solution only if the following the solvability condition is satisfied:

$$-\iint\limits_\omega \Delta_\eta U_2(\eta, z)\, d\eta + \int\limits_\Gamma \frac{\partial U_2}{\partial \nu}(\eta, z)\, ds_\eta = 0.$$

Substituting the right-hand sides of the relations (4.10) into the equation above, we arrive at the following equation for the leading asymptotic term:

$$|\omega| \frac{d^2 U_0}{dz^2} + \varkappa^{-1} \iint\limits_\omega f(\varepsilon\eta, z)\, d\eta - k_0 |\Gamma| U_0 = 0. \tag{4.11}$$

Here, $|\omega|$ and $|\Gamma|$ is the area and perimeter of ω.

Returning to the original variables by formula (4.5) and taking the notation (4.4) into account, we can rewrite Eq. (4.11) as follows:

$$-\frac{d^2 U_0}{dz^2} + \frac{|\Gamma_\varepsilon|}{|\omega_\varepsilon|} k U_0 = \varkappa^{-1} F(z). \tag{4.12}$$

Here, $|\omega_\varepsilon|$ and $|\Gamma_\varepsilon|$ are the area and perimeter of the cross-section ω_ε,

$$F(z) = \frac{1}{|\omega_\varepsilon|} \iint\limits_{\omega_\varepsilon} f(y, z)\, dy.$$

The function $U_0(z)$, generally speaking, does not satisfy the boundary conditions (4.3). In the vicinities of the rod ends, where the phenomenon of boundary layer arises, the inner asymptotic expansion should be constructed (see, for example, [23], Ch. 2, § 1). As a result, Eq. (4.12) is subjected to the following boundary conditions:

$$U_0(\pm l) = \frac{1}{|\omega_\varepsilon|} \iint\limits_{\omega_\varepsilon} u_0^{\pm}(y)\, dy. \tag{4.13}$$

Note that in the right-hand side of the relation (4.13), there appears the arithmetic mean value of the right-hand side of the boundary condition (4.3).

The resulting equation (4.12) represents the well-known equation of the one-dimensional theory of heat-conduction in a thin rod.

3.5 Asymptotic modeling of the heat conduction in a thin plate

3.5.1 Heat-conduction in a thin plate

We consider the problem of stationary heat-conduction in a thin plate Ω_ε of the thickness $2\varepsilon h$ with the middle-section ω bounded by the contour Γ (see Fig. 5.2). Here, ε is a small positive parameter, which is equal to the ratio of the plate half-thickness $h_\varepsilon = \varepsilon h$ to the characteristic size of the plate middle-section. In other words, the plate Ω_ε is obtained from the cylinder $\omega \times (-h, h)$ by compressing in the longitudinal direction by ε^{-1} times, i. e.,

$$\Omega_\varepsilon = \bigl\{ x = (y, z) : \ y \in \omega, \ \varepsilon^{-1} z \in (-h, h) \bigr\}.$$

According to Fourier's law, the temperature distribution $u(x)$ in the inner points of the plate is described the equation

$$\frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2} + \frac{\partial^2 u}{\partial z^2} + \varkappa^{-1} f(x) = 0, \tag{5.1}$$

where $f(x)$ is the density of volume heat sources, $\varkappa$ is the heat conductivity.

Let us assume that the face surfaces of the plate

$$\Sigma_\varepsilon^{\pm} = \bigl\{ x : \ y \in \omega, \ z = \pm \varepsilon h \bigr\}$$

exchange heat with the surrounding medium according to Newton's law

$$-\frac{\partial u}{\partial z} = ku - \varkappa^{-1} q^{+}, \quad z = \varepsilon h; \tag{5.2}$$

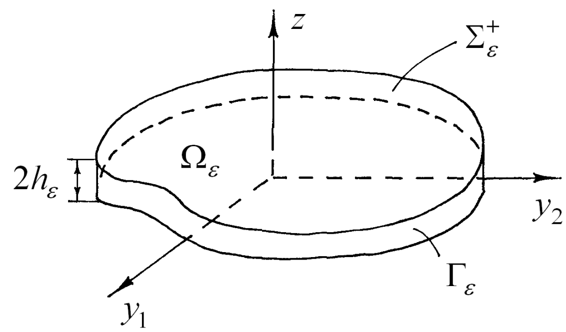

Figure 5.2: The plate geometry and the coordinate system.

$$\frac{\partial u}{\partial z} = ku - \varkappa^{-1} q^-, \quad z = \varepsilon h. \tag{5.3}$$

Here the quantities $q^+(y)$ and $q^-(y)$ are the projections of the heat flux on the outer normal taken with the inverse sign.

To fix our ideas, we assume that

$$u\big|_{\Gamma_\varepsilon} = u_0(y, z), \quad y \in \Gamma, \ z \in (-h_\varepsilon, h_\varepsilon), \tag{5.4}$$

that is the lateral surface of the plate $\Gamma_\varepsilon = \big\{x : \ y \in \Gamma, \ \varepsilon^{-1} z \in (-h, h)\big\}$ is maintained at the prescribed temperature.

3.5.2 Quasi two-dimensional process of heat-conduction in a thin plate

Following Zino and Tropp (see, [23], Ch. IV, § 1), we will construct an approximate (for $\varepsilon \ll 1$) solution to the problem (5.1) – (5.4) under the smallness assumption on the heat exchange coefficient

$$k = \varepsilon k_0. \tag{5.5}$$

Moreover, in order to equalize the orders of terms in the right-hand sides of the boundary conditions (5.2) and (5.3), we put

$$q^\pm(y) = \varepsilon q_0^\pm(y). \tag{5.6}$$

Introducing in Eq. (5.1) and the boundary conditions (5.2), (5.3) the stretched variable

$$\zeta = \varepsilon^{-1} z, \tag{5.7}$$

we obtain

$$\frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2} + \varepsilon^{-2}\frac{\partial^2 u}{\partial \zeta^2} + \varkappa^{-1} f(y, \varepsilon\zeta) = 0. \tag{5.8}$$

Moreover, tacking into account the assumptions (5.5) and (5.6), we get

$$\mp\varepsilon^{-1}\frac{\partial u}{\partial \zeta} = \varepsilon k_0 u - \varepsilon\varkappa^{-1} q_0^{\pm}, \quad \zeta = \pm h. \tag{5.9}$$

Far from the lateral surface of the plate, the solution to the problem (5.8), (5.9), (5.4) will be constructed in the form of asymptotic expansion

$$u(x) = U_0(y,\zeta) + \varepsilon^2 U_2(y,\zeta) + \ldots\ . \tag{5.10}$$

After substituting the expansion (5.10) into the relations (5.8), (5.9) and collecting terms, we equate to zero the coefficients at subsequent powers of ε. As a result, we derive the following relations:

$$\frac{\partial^2 U_0}{\partial \zeta^2}(y,\zeta) = 0, \quad \zeta \in (-h,h); \qquad \frac{\partial U_0}{\partial \zeta}(y,\pm h) = 0; \tag{5.11}$$

$$-\frac{\partial^2 U_2}{\partial \zeta^2}(y,\zeta) = \Delta_y U_0 + \varkappa^{-1} f, \quad \zeta \in (-h,h); \tag{5.12}$$

$$\mp\frac{\partial U_2}{\partial \zeta}(y,\pm h) = k_0 U_0(y,\pm h) - \varkappa^{-1} q_0^{\pm}(y). \tag{5.13}$$

It is evident that the solution to the homogeneous problem (5.11) is an arbitrary function $U_0(y)$ of the transverse coordinates $y = (y_1, y_2)$. The problem (5.12), (5.13) represents an analogue of the nonhomogeneous Neumann problem, and, therefore, it is solvable only under a certain special condition.

Following [23] and integrating Eq. (5.12), we get

$$-\frac{\partial U_2}{\partial \zeta}(y,\zeta) = -\frac{\partial U_2}{\partial \zeta}(y,-h) + \int\limits_{-h}^{\zeta} \bigl(\Delta_y U_0(y) + \varkappa^{-1} f(y,\varepsilon t)\bigr)\, dt.$$

Taking into account the boundary condition (5.13) for $\zeta = -h$, we obtain

$$-\frac{\partial U_2}{\partial \zeta}(y,\zeta) = -k_0 U_0(y) + \varkappa^{-1} q_0^-(y) + (\zeta + h)\Delta_y U_0(y) + \varkappa^{-1} \int\limits_{-h}^{\zeta} f(y, \varepsilon t)\, dt.$$

Now, satisfying the boundary condition (5.13) for $\zeta = h$, we find

$$-2h\Delta_y U_0(y) + 2k_0 U_0(y) = \varkappa^{-1} \left(\int\limits_{-h}^{h} f(y, \varepsilon\zeta)\, d\zeta + q_0^+(y) + q_0^-(y) \right). \quad (5.14)$$

If the solvability condition (5.14) for the problem (5.12), (5.13) is satisfied, then its solution exists and is determined up to an arbitrary function of the transverse coordinates. On the other hand, the solvability condition (5.14) represents the sought-for equation for determining the leading asymptotic term.

Returning to the original longitudinal coordinate by formula (5.7) and taking into account the notation (5.5) and (5.6), we rewrite Eq. (5.14) in its final form as follows:

$$-\Delta_y U_0 + \frac{k}{h_\varepsilon} U_0 = \varkappa^{-1} F(y). \quad (5.15)$$

Here, $h_\varepsilon = \varepsilon h$ is the half-width of the plate Ω_ε, and $F(y)$ is given by

$$F(y) = \frac{1}{2h_\varepsilon} \int\limits_{-h_\varepsilon}^{h_\varepsilon} f(y, z)\, dz + \frac{1}{2h_\varepsilon} \big[q_0^+(y) + q_0^-(y) \big].$$

The function $U_0(y)$, generally speaking, does not satisfy the boundary condition (5.4) due to its dependence on the longitudinal coordinate. In the vicinity of the plate lateral surface, there arises the phenomenon of a boundary layer. It can be shown that Eq. (5.15) should be subjected to the following boundary condition resulting from the boundary condition (5.4):

$$U_0(y) = \frac{1}{2h_\varepsilon} \int\limits_{-h_\varepsilon}^{h_\varepsilon} u_0(y, z)\, dz, \qquad y \in \Gamma. \quad (5.16)$$

The resulting equation (5.15) together with the boundary condition (5.16) allows to determine the function $U_0(y)$, which is the thickness-average temperature of the plate in the problem (5.1)–(5.4) in the first approximation.

3.6 Method of matched asymptotic expansions

3.6.1 Thermal contact conductivity of a cluster of microcontacts

Consider two bodies which are in contact at nominally flat surfaces touching each other only at few discrete spots $\omega_1, \dots, \omega_N$. Assume that the following conditions are satisfied: 1) the bodies are placed in a vacuum; 2) heat flows normally to the contact interface from one body to the other; 3) boundary condition of uniform temperature is imposed across the contact area [12]. Then, we can describe the temperature distribution near the contact interface in terms of a harmonic potential $u(\mathbf{x})$ as follows:

$$\begin{aligned} T(\mathbf{x}) &= T_1 + \frac{k_2(T_2 - T_1)}{k_1 + k_2} u(\mathbf{x}), \quad x_3 > 0, \\ &= T_2 - \frac{k_1(T_2 - T_1)}{k_1 + k_2} u(x_1, x_2, -x_3), \quad x_3 < 0. \end{aligned}$$

Here, k_1 and k_2 are the thermal conductivities of the bodies.

The potential $u(\mathbf{x})$ is the unique solution of the following boundary-value problem:

$$\Delta_x u(\mathbf{x}) = 0, \quad x_3 > 0; \tag{6.1}$$

$$u(x_1, x_2, 0) = 1, \quad (x_1, x_2) \in \bigcup_{j=1}^{N} \omega_j; \tag{6.2}$$

$$\frac{\partial u}{\partial x_3}(x_1, x_2, 0) = 0, \quad (x_1, x_2) \notin \bigcup_{j=1}^{N} \overline{\omega}_j; \tag{6.3}$$

$$u(\mathbf{x}) = o(1), \quad |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} \to \infty. \tag{6.4}$$

The total heat flow conducted between the two bodies is

$$Q = -\sum_{j=1}^{N} \frac{k_1 k_2 (T_2 - T_1)}{k_1 + k_2} \iint\limits_{\omega_j} \frac{\partial u}{\partial x_3}(x_1, x_2, +0)\, dx_1 dx_2, \tag{6.5}$$

and it is found to be proportional to the temperature drop $T_2 - T_1$. The constant of proportionality defined by

$$h_c = \frac{Q}{(T_2 - T_1)A_a}, \tag{6.6}$$

where A_a is the area of apparent contact, is known as the thermal interfacial contact conductance, abbreviated herein to contact conductance [12]. Notice also that the quantity Q/A_a determines the heat flux across the area A_a.

3.6.2 Governing integral equation

The solution of the problem (6.1) – (6.4) can be represented in the form of a single-layer potential

$$u(\mathbf{x}) = \frac{1}{2\pi} \sum_{j=1}^{N} \iint\limits_{\omega_j} \frac{\mu_j(y_1, y_2)\, dy_1 dy_2}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2}}. \tag{6.7}$$

Accordingly, the limit value of its normal derivative is given by the following formula ($j = 1, \ldots, N$):

$$\frac{\partial u}{\partial x_3}(x_1, x_2, +0) = -\mu_j(x_1, x_2), \quad (x_1, x_2) \in \omega_j. \tag{6.8}$$

Due to the boundary condition (6.2), the integrand density functions $\mu_1(x_1, x_2), \ldots, \mu_N(x_1, x_2)$ satisfy the following system of Fredholm integral equations of the first kind ($j = 1, \ldots, N$):

$$\big(\mathcal{B}^j \mu_j\big)(x_1, x_2) + \sum_{k \neq j} \big(\mathcal{B}^k \mu_k\big)(x_1, x_2) = 1, \quad (x_1, x_2) \in \omega_j. \tag{6.9}$$

Here, $\mathcal{B}^k$ is an integral operator defined by the formula

$$\big(\mathcal{B}^k \mu_k\big)(x_1, x_2) = \frac{1}{2\pi} \iint\limits_{\omega_k} \frac{\mu_k(\mathbf{y})\, d\mathbf{y}}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}}. \tag{6.10}$$

Thus, from (6.5) and (6.6), it follows that

$$h_c A_a = \frac{k_1 k_2}{k_1 + k_2} \sum_{j=1}^{N} \iint\limits_{\omega_j} \mu_j(\mathbf{y})\, d\mathbf{y} \tag{6.11}$$

Notice that the apparent area of contact for a cluster of microcontacts is a notion without mathematically precise definition. That is why formula (6.19) determines the contact conductance h_c with accuracy up to determining the area A_a of apparent contact.

3.6.3 Capacity of a contact spot

Consider the potential for a single contact spot problem

$$u_j^0(\mathbf{x}) = \frac{1}{2\pi} \iint\limits_{\omega_j} \frac{\mu_j^0(\mathbf{y})\, d\mathbf{y}}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2}}, \tag{6.12}$$

where $\mu_j^0(x_1, x_2)$ is the solution of the integral equation

$$\left(\mathcal{B}^j \mu_j^0\right)(x_1, x_2) = 1, \quad (x_1, x_2) \in \omega_j. \tag{6.13}$$

As $|\mathbf{x}| \to \infty$, the following asymptotic formula holds:

$$u_j^0(\mathbf{x}) = \frac{\mathbf{c}_j}{|\mathbf{x}|} + O(|\mathbf{x}|^{-2}). \tag{6.14}$$

Here, $\mathbf{c}_j$ is the harmonic capacity [20]. Notice that $\mathbf{c}_j$ coincides with the electrostatic capacity of the cylindrical conductor of electricity of infinitesimal height shaped as ω_j.

From (6.12) and (6.14), it follows that

$$\mathbf{c}_j = \frac{1}{2\pi} \iint\limits_{\omega_j} \mu_j^0(\mathbf{y})\, d\mathbf{y}. \tag{6.15}$$

Let us now introduce the center, P^j, of the contact spot ω_j as the center of mass of the domain ω_j with the mass density $\mu_j^0(x_1, x_2)$. In other words, the coordinates of the point P^j are given by

$$x_i^j = \frac{\displaystyle\iint\limits_{\omega_j} y_i \mu_j^0(\mathbf{y})\, d\mathbf{y}}{\displaystyle\iint\limits_{\omega_j} \mu_j^0(\mathbf{y})\, d\mathbf{y}} \quad (i = 1, 2). \tag{6.16}$$

Observe that in accordance with Eq. (6.13) and the maximum principle for harmonic functions, we have

$$\mu_j^0(x_1, x_2) > 0, \quad (x_1, x_2) \in \omega_j \quad (j = 1, \dots, N).$$

Notice also that the same inequality takes place for the densities $\mu_j(x_1, x_2)$ $(j = 1, \dots, N)$.

Finally, from (6.16) it immediately follows that

$$\iint\limits_{\omega_j} (y_i - x_i^j)\mu_j^0(\mathbf{y})\, d\mathbf{y} = 0 \quad (i = 1, 2;\ j = 1, \dots, N).$$

This implies that if we choose point P^j as the origin, then we will have the following asymptotic formula (compare with (6.14)): $u_j^0(\mathbf{x}) = \mathbf{c}_j|\mathbf{x}|^{-1} + O(|\mathbf{x}|^{-3})$ as $|\mathbf{x}| \to \infty$.

3.6.4 Capacity of a cluster of contact spots

By analogy with formula (6.14), we write out the following asymptotic expansion of the potential (6.7) at infinity:

$$u(\mathbf{x}) = \frac{\mathbf{C}}{|\mathbf{x}|} + O(|\mathbf{x}|^{-2}), \quad |\mathbf{x}| \to \infty. \tag{6.17}$$

For brevity, the constant $\mathbf{C}$ will be called the harmonic capacity of the cluster $\bigcup_{j=1}^N \omega_j$.

From (6.7) and (6.17), it follows that

$$\mathbf{C} = \frac{1}{2\pi} \sum_{j=1}^{N} \iint\limits_{\omega_j} \mu_j(\mathbf{y})\, d\mathbf{y}. \tag{6.18}$$

In view of (6.18), formula (6.11) for the thermal contact conductivity takes the form

$$h_c = \frac{2\pi k_1 k_2}{(k_1 + k_2)A_a}\, \mathbf{C}, \tag{6.19}$$

where $\mathbf{C}$ is the capacity of the cluster.

Formula (6.19) determines the thermal contact conductivity of a cluster in terms of its harmonic capacity.

Notice that by analogy with formula (6.16), the notion of the center of cluster can be introduced. Such a generalization will be required for describing the interaction between clusters of microcontacts.

3.6.5 Singularly perturbed boundary value problem

Denote by d the minimum possible distance $d_{jk} = |P^j - P^k|$, $j \neq k$. Let ω_1^j be a simply connected domain on a plane that contains the coordinate origin and is enclosed in a circle of diameter d with the center at the origin. Introducing a small positive parameter ε, we set

$$\omega_\varepsilon^j = \big\{(x_1, x_2) : \ \varepsilon^{-1}(x_1 - x_1^j, x_2 - x_2^j) \in \omega_1^j\big\}.$$

Hence, assuming that $\varepsilon \in (0, 1]$, we will have that each contact spot ω_ε^j is completely contained in the circle with center P^j and radius $d/2$. Thus, the contact spots $\omega_\varepsilon^1, \dots, \omega_\varepsilon^N$ are not intersecting.

Let us consider the following boundary-value problem:

$$\Delta_x u^\varepsilon(\mathbf{x}) = 0, \quad x_3 > 0; \tag{6.20}$$

$$u^\varepsilon(x_1, x_2, 0) = g_0^j, \quad (x_1, x_2) \in \omega_\varepsilon^j, \quad j = 1, 2, \dots, N; \tag{6.21}$$

$$\frac{\partial u^\varepsilon}{\partial x_3}(x_1, x_2, 0) = 0, \quad (x_1, x_2) \notin \bigcup_{j=1}^{N} \overline{\omega_\varepsilon^j}; \tag{6.22}$$

$$u^\varepsilon(\mathbf{x}) = o(1), \quad |\mathbf{x}| \to \infty. \tag{6.23}$$

For the sake of simplicity, we assume that g_0^j $(j = 1, 2, \dots, N)$ are constants.

We use the method of matched asymptotic expansions to construct the leading asymptotic terms of the outer (which is valid far from the points $P^1, \dots, P^N$) and the inner (near the contact spots $\omega_\varepsilon^1, \dots, \omega_\varepsilon^N$) asymptotic expansions of the solution $u^\varepsilon(\mathbf{x})$ to the singularly perturbed boundary value problem (6.20)–(6.23).

3.6.6 First limit problem. Outer asymptotic expansion

Passing to the limit as $\varepsilon \to 0$ in (6.20)–(6.23), we obtain that the Dirichlet boundary conditions (6.21) disappear, and we arrive at the following first limit problem

$$\Delta_x v(\mathbf{x}) = 0, \quad x_3 > 0; \tag{6.24}$$

$$\frac{\partial v}{\partial x_3}(x_1, x_2, 0) = 0, \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{P^1, \dots, P^N\}; \tag{6.25}$$

$$v(\mathbf{x}) = o(1), \quad |\mathbf{x}| \to \infty. \tag{6.26}$$

The solution to the homogeneous problem (6.24) – (6.26) is sought in the form of expansion

$$v(\mathbf{x}) = \nu_1(\varepsilon)v_1(\mathbf{x}) + \nu_2(\varepsilon)v_2(\mathbf{x}) + \ldots, \tag{6.27}$$

where $\{\nu_q(\varepsilon)\}$ is an asymptotic sequence of gauge functions.

Recall that a sequence $\nu_q(\varepsilon)$, $q = 1, 2, \ldots$, is called *asymptotic sequence*, if the functions $\nu_q(\varepsilon)$ are defined and positive in a vicinity $(0, \varepsilon_0)$ of the point 0 and for every q the asymptotic condition $\nu_{q+1}(\varepsilon)/\nu_q(\varepsilon) \to 0$ is satisfied as $\varepsilon \to 0$ for every $q = 1, 2, \ldots$.

Substituting the expansion (6.27) into Eqs. (6.24) – (6.26), we find that the function $v_q(\mathbf{x})$, $q = 1, 2, \ldots$, should satisfy the following problem:

$$\Delta_x v_q(\mathbf{x}) = 0, \quad x_3 > 0; \tag{6.28}$$

$$\frac{\partial v_q}{\partial x_3}(x_1, x_2, 0) = 0, \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{P^1, \ldots, P^N\}; \tag{6.29}$$

$$v_q(\mathbf{x}) = o(1), \quad |\mathbf{x}| \to \infty. \tag{6.30}$$

It is clear that all nontrivial solutions to the problem (6.28) – (6.30) should have singularities as $\mathbf{x} \to P^j$, $j = 1, 2, \ldots, N$. Moreover, the functions $v_q(\mathbf{x})$, $q = 1, 2, \ldots$, are linear combinations of the fundamental solution and its derivatives.

A description of the behavior of functions $v_q(\mathbf{x})$ at the points $P^1, \ldots, P^N$ is possible only after performing the asymptotic procedure of matching of the expansion (6.26) with the inner asymptotic expansion.

3.6.7 Second limit problem. Inner asymptotic expansion

To construct the inner asymptotic expansion of the solution $u^\varepsilon(\mathbf{x})$, we pass to the "stretched" coordinates $\boldsymbol{\xi}^j = (\xi_1^j, \xi_2^j, \xi_3^j)$ in the vicinity of the individual contact spot ω_ε^j as follows:

$$\boldsymbol{\xi}^j = \varepsilon^{-1}(\mathbf{x} - P^j) = \varepsilon^{-1}(x_1 - x_1^j, x_2 - x_2^j, x_3). \tag{6.31}$$

The inner asymptotic expansion will be constructed in the form

$$u^\varepsilon(\mathbf{x}) = \mu_0(\varepsilon)w_0^j(\boldsymbol{\xi}^j) + \mu_1(\varepsilon)w_1^j(\boldsymbol{\xi}^j) + \ldots . \tag{6.32}$$

The functions $w_r^j(\boldsymbol{\xi}^j)$ are solutions to the following second limit problem ($j = 1, 2, \dots, N$):

$$\Delta_\xi w_r^j(\boldsymbol{\xi}) = 0, \ \xi_3 > 0; \tag{6.33}$$

$$\mu_0(\varepsilon) w_0^j(\xi_1, \xi_2, 0) = g_0^j, \quad (\xi_1, \xi_2) \in \omega_1^j; \tag{6.34}$$

$$w_r^j(\xi_1, \xi_2, 0) = 0, \quad (\xi_1, \xi_2) \in \omega_1^j, \quad (r \neq 0); \tag{6.35}$$

$$\frac{\partial w_r^j}{\partial \xi_3}(\xi_1, \xi_2, 0) = 0, \quad (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \overline{\omega_1^j}. \tag{6.36}$$

Observe that in the stretched coordinates (6.31), the distance from the center of the fixed contact spot ω_ε^j to other contact spots becomes of order $\varepsilon^{-1} d/2$, while the representation of the domain ω_1^j does not depend on the parameter ε. That is why, passing to the limit as $\varepsilon \to 0$, we obtain the second limit problem in the semi-infinite domain for a single contact ω_1^j.

Further, Eqs. (6.33) – (6.36) also do not determine unambiguously the functions $w_r^j(\boldsymbol{\xi}^j)$, if we admit that they possess a growth at the infinity.

Let us emphasize that, generally speaking, the structure of the asymptotic sequences $\nu_k(\varepsilon)$ and $\mu_j(\varepsilon)$ is not clear a priori and must be determined in the process of constructing the asymptotics.

3.6.8 Leading asymptotic term of the inner expansion

We start with the first non-trivial limit problem (6.33), (6.34), (6.36). Since the right-hand side of (6.34) does not depend on ε, we put

$$\mu_0(\varepsilon) = 1. \tag{6.37}$$

In view of the asymptotic condition (6.23), we impose the following asymptotic condition:

$$w_0^j(\boldsymbol{\xi}) = o(1), \quad |\boldsymbol{\xi}| \to \infty. \tag{6.38}$$

The unique solution to the problem (6.33), (6.34), (6.36), (6.38) is given by

$$w_0^j(\boldsymbol{\xi}) = Y_j(\boldsymbol{\xi}), \quad j = 1, 2, \dots, N,$$

$$Y_j(\boldsymbol{\xi}) = \frac{1}{2\pi} \iint\limits_{\omega_1^j} \frac{\chi_j^0(\boldsymbol{\eta})\, d\boldsymbol{\eta}}{\sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + \xi_3^2}},$$

where $\chi_j^0(\xi_1, \xi_2)$ is the unique solution of the integral equation

$$\frac{1}{2\pi} \iint\limits_{\omega_1^j} \frac{\chi_j^0(\boldsymbol{\eta})\, d\boldsymbol{\eta}}{\sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}} = 1, \quad (\xi_1, \xi_2) \in \omega_1^j.$$

Making use of the asymptotic formula $(1+x)^{-1/2} = 1 - (1/2)x + O(x^2)$ as $x \to 0$, we obtain

$$\frac{1}{\sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + \xi_3^2}} = \frac{1}{|\boldsymbol{\xi}|} + \frac{\xi_1\eta_1 + \xi_2\eta_2}{|\boldsymbol{\xi}|^3} + O(|\boldsymbol{\xi}|^{-3}).$$

Hence, the following asymptotic expansion of the function $w_0^j(\boldsymbol{\xi})$ as $|\boldsymbol{\xi}| \to \infty$ holds true:

$$w_0^j(\boldsymbol{\xi}) = \frac{f_0^j}{2\pi|\boldsymbol{\xi}|} + \frac{m_1^j\xi_1 + m_2^j\xi_2}{2\pi|\boldsymbol{\xi}|^3} + O(|\boldsymbol{\xi}|^{-3}). \tag{6.39}$$

Here we introduced the notation

$$f_0^j = \iint\limits_{\omega_1^j} \chi_j^0(\boldsymbol{\eta})\, d\boldsymbol{\eta}, \quad m_i^j = \iint\limits_{\omega_1^j} \eta_i \chi_j^0(\boldsymbol{\eta})\, d\boldsymbol{\eta}, \quad (i = 1, 2). \tag{6.40}$$

Note that in view of (6.14), we have

$$f_0^j = 2\pi \mathbf{c}_1^j, \tag{6.41}$$

where $\mathbf{c}_1^j$ is the harmonic capacity of the disk $\{\boldsymbol{\xi} : (\xi_1, \xi_2) \in \overline{\omega}_1^j,\ \xi_3 = 0\}$. Note also that if the point P^j coincides with the center of the contact spot ω_ε^j, then in view of (6.16), we will have $m_1^j = m_2^j = 0$. Thus, the appropriate choice of the coordinate system simplifies the asymptotic constructions.

3.6.9 Leading asymptotic term of the outer expansion

Changing the stretched coordinates in (6.39) according to (6.31), we obtain

$$w_0^j(\boldsymbol{\xi}) = \varepsilon \frac{f_0^j}{2\pi|\mathbf{x} - P^j|} + \varepsilon^2 \frac{m_1^j(x_1 - x_1^j) + m_2^j(x_2 - x_2^j)}{2\pi|\mathbf{x} - P^j|^3} + O(\varepsilon^3|\mathbf{x} - P^j|^{-3}), \tag{6.42}$$

The asymptotic expansion (6.42) implies that

$$\nu_q(\varepsilon) = \varepsilon^q, \quad q = 1, 2, \ldots . \tag{6.43}$$

Furthermore, the leading asymptotic term of formula (6.42) yields the following asymptotic conditions for the leading term of the outer asymptotic expansion (6.27):

$$v^1(\mathbf{x}) = \frac{f_0^j}{2\pi|\mathbf{x} - P^j|} + O(1), \quad \mathbf{x} \to P^j. \tag{6.44}$$

The asymptotic formula (6.44) determines the order of singularity at the points $P^1, \ldots, P^N$.

The solution of the problem (6.28) – (6.30), (6.44) is sought in the form of a linear combination

$$v^1(\mathbf{x}) = \sum_{j=1}^{N} c_1^j \Phi(\mathbf{x} - P^j), \tag{6.45}$$

where $\Phi(\mathbf{x})$ is a singular fundamental solution of the Laplace equation in the half-space $x_3 \geq 0$ with a pole on its surface $x_3 = 0$, i. e.,

$$\Phi(\mathbf{x}) = \frac{1}{2\pi|\mathbf{x} - P^j|}.$$

Taking into account the asymptotic conditions (6.44) and Eqs. (6.41), we immediately find that

$$c_1^j = f_0^j = 2\pi\mathbf{c}_1^j, \tag{6.46}$$

where $\mathbf{c}_1^j$ is determined by Eq. (6.15).

3.6.10 Second-order asymptotic terms

From (6.45) it follows that

$$v^1(\mathbf{x}) = c_1^j \Phi(\mathbf{x} - P^j) + \varphi(P^j) + O(|\mathbf{x} - P^j|), \tag{6.47}$$

where we introduced the notation

$$\varphi(P^j) = \sum_{k \neq j} c_1^k \Phi(\mathbf{x} - P^j). \tag{6.48}$$

Making change of the variables (6.31) in (6.47), we obtain

$$v^1(\mathbf{x}) = \varepsilon^{-1} c_1^j \Phi(\boldsymbol{\xi}^j) + \varphi(P^j) + O(\varepsilon|\boldsymbol{\xi}^j|),$$

and, consequently,

$$\varepsilon v^1(\mathbf{x}) = c_1^j \Phi(\boldsymbol{\xi}^j) + \varepsilon\varphi(P^j) + O(\varepsilon^2|\boldsymbol{\xi}^j|), \tag{6.49}$$

Thus, in view of (6.27), (6.32), and (6.49), we find that

$$\mu_r(\varepsilon) = \varepsilon^r, \quad r = 0, 1, 2, \ldots . \tag{6.50}$$

Furthermore, we obtain the following matching asymptotic conditions ($j = 1, 2, \ldots, N$):

$$w_1^j(\boldsymbol{\xi}) = \varphi(P^j) + O(|\boldsymbol{\xi}|^{-1}), \quad |\boldsymbol{\xi}| \to \infty. \tag{6.51}$$

The solution to the problem (6.33), (6.35), (6.36), (6.51) is given by

$$w_1^j(\boldsymbol{\xi}) = \varphi(P^j)\big(1 - Y_j(\boldsymbol{\xi})\big), \quad j = 1, 2, \ldots, N. \tag{6.52}$$

In order to construct the second term of the outer asymptotic expansion (6.27), we need a two-term asymptotic representation of $w_0^j(\boldsymbol{\xi})$ as $|\boldsymbol{\xi}| \to \infty$ and a one-term asymptotic representation of $w_1^j(\boldsymbol{\xi})$. In other words, we will make use of the following asymptotic formulas (compare with (6.39)):

$$w_0^j(\boldsymbol{\xi}) = f_0^j \Phi(\boldsymbol{\xi}) - \sum_{i=1}^{2} m_i^j \frac{\partial \Phi}{\partial \xi_i}(\boldsymbol{\xi}) + O(|\boldsymbol{\xi}|^{-3}), \quad |\boldsymbol{\xi}| \to \infty, \tag{6.53}$$

$$w_1^j(\boldsymbol{\xi}) = \varphi(P^j) + f_1^j \Phi(\boldsymbol{\xi}) + O(|\boldsymbol{\xi}|^{-2}), \quad |\boldsymbol{\xi}| \to \infty. \tag{6.54}$$

Here we introduced the notation (see (6.52))

$$f_1^j = -2\pi \mathbf{c}_1^j \varphi(P^j). \tag{6.55}$$

Changing the stretched coordinates in (6.53) and (6.54) according to (6.31), we obtain

$$\begin{aligned} w_0^j(\boldsymbol{\xi}) + \varepsilon w_1^j(\boldsymbol{\xi}) = {} & \varepsilon\big(f_0^j \Phi(\mathbf{x} - P^j) + \varphi(P^j)\big) \\ & + \varepsilon^2 \Big\{ -\sum_{i=1}^{2} m_i^j \frac{\partial \Phi}{\partial x_i}(\mathbf{x} - P^j) + f_1^j \Phi(\mathbf{x} - P^j) \Big\} \\ & + O(\varepsilon^3 |\mathbf{x} - P^j|^{-3}). \end{aligned}$$

Thus, in view of (6.27) and (6.43), the obtained formula implies the following asymptotic condition:

$$v^2(\mathbf{x}) = -\sum_{i=1}^{2} m_i^j \frac{\partial \Phi}{\partial x_i}(\mathbf{x} - P^j) + f_1^j \Phi(\mathbf{x} - P^j) + O(1), \quad \mathbf{x} \to P^j. \tag{6.56}$$

It is easy to see that the order of singularity of terms $v^q(\mathbf{x})$ in the outer asymptotic expansion (6.27) at the points $P^1, \ldots, P^N$ increases with the number q.

The singular part of the asymptotic expansion (6.56) uniquely determines the function $v^2(\mathbf{x})$, that is

$$v^2(\mathbf{x}) = -\sum_{i=1}^{2} m_i^j \frac{\partial \Phi}{\partial x_i}(\mathbf{x} - P^j) + f_1^j \Phi(\mathbf{x} - P^j), \tag{6.57}$$

where the coefficients m_i^j, $i = 1, 2$, and f_1^j are given by (6.40) and (6.55).

3.6.11 Asymptotic matching procedure

Let us consider the two-term asymptotic representations

$$u^\varepsilon(\mathbf{x}) \simeq \varepsilon v_1(\mathbf{x}) + \varepsilon^2 v_2(\mathbf{x}), \tag{6.58}$$

$$u^\varepsilon(\mathbf{x}) \simeq w_0^j(\boldsymbol{\xi}^j) + \varepsilon w_1^j(\boldsymbol{\xi}^j), \quad j = 1, 2, \ldots, N. \tag{6.59}$$

According to (6.45) and (6.57), we have

$$\varepsilon v_1(\mathbf{x}) + \varepsilon^2 v_2(\mathbf{x}) = \varepsilon \sum_{j=1}^{N} f_0^j \Phi(\mathbf{x} - P^j) + \varepsilon^2 \sum_{j=1}^{N} \biggl\{ f_1^j \Phi(\mathbf{x} - P^j) - \sum_{i=1}^{2} m_i^j \frac{\partial \Phi}{\partial x_i}(\mathbf{x} - P^j) \biggr\}.$$

Introducing the stretched coordinates (6.31) into the right-hand side of this relation and taking into account the formulas

$$\Phi(\mathbf{x} - P^j) = \Phi(\varepsilon \boldsymbol{\xi}^j) = \varepsilon^{-1} \Phi(\boldsymbol{\xi}^j),$$

$$\frac{\partial \Phi}{\partial x_i}(\mathbf{x} - P^j) = \varepsilon^{-2} \frac{\partial \Phi}{\partial \xi_i^j}(\boldsymbol{\xi}^j),$$

we obtain

$$\begin{aligned}\varepsilon v_1(\mathbf{x}) + \varepsilon^2 v_2(\mathbf{x}) = {} & f_0^j \Phi(\mathbf{x} - P^j) + \varepsilon \sum_{k \neq j} f_0^k \Phi(P^j - P^k + \varepsilon \boldsymbol{\xi}^j) \\ & + \varepsilon^2 \left(-\varepsilon^{-2} \sum_{i=1}^{2} m_i^j \frac{\partial \Phi}{\partial \xi_i}(\boldsymbol{\xi}^j) + \varepsilon^{-1} f_1^j \Phi(\boldsymbol{\xi}^j) \right) \\ & + \varepsilon^2 \sum_{k \neq j} \left\{ \sum_{i=1}^{2} m_i^j \frac{\partial \Phi}{\partial x_i^k}(P^j - P^k + \varepsilon \boldsymbol{\xi}^j) \right. \\ & \left. + f_1^k \Phi(P^j - P^k + \varepsilon \boldsymbol{\xi}^j) \right\}. \end{aligned} \tag{6.60}$$

On the other hand, in view of (6.53) and (6.54), we have

$$\begin{aligned} w_0^j(\boldsymbol{\xi}^j) + \varepsilon w_1^j(\boldsymbol{\xi}^j) = {} & f_0^j \Phi(\mathbf{x} - P^j) - \sum_{i=1}^{2} m_i^j \frac{\partial \Phi}{\partial \xi_i}(\boldsymbol{\xi}^j) \\ & + \varepsilon \big(\varphi(P^j) + f_1^j \Phi(\boldsymbol{\xi}^j) \big) \\ & + O(|\boldsymbol{\xi}^j|^{-3}) + O(\varepsilon |\boldsymbol{\xi}^j|^{-2}), \quad |\boldsymbol{\xi}^j| \to \infty. \end{aligned} \tag{6.61}$$

Further, taking into account (6.46), and(6.48), we can rewrite the asymptotic relations (6.60) and (6.61) as follows:

$$\varepsilon v_1(\mathbf{x}) + \varepsilon^2 v_2(\mathbf{x}) = \mathfrak{A}(\varepsilon; \boldsymbol{\xi}^j) + O(\varepsilon^2 |\boldsymbol{\xi}^j|) \tag{6.62}$$

$$w_0^j(\boldsymbol{\xi}^j) + \varepsilon w_1^j(\boldsymbol{\xi}^j) = \mathfrak{A}(\varepsilon; \boldsymbol{\xi}^j) + O(|\boldsymbol{\xi}^j|^{-3}) + O(\varepsilon |\boldsymbol{\xi}^j|^{-2}). \tag{6.63}$$

Here, $\mathfrak{A}(\varepsilon; \boldsymbol{\xi}^j)$ denotes the matched asymptotic terms, i. e.,

$$\mathfrak{A}(\varepsilon; \boldsymbol{\xi}^j) = f_0^j \Phi(\mathbf{x} - P^j) - \sum_{i=1}^{2} m_i^j \frac{\partial \Phi}{\partial \xi_i}(\boldsymbol{\xi}^j) + \varepsilon \big(\varphi(P^j) + f_1^j \Phi(\boldsymbol{\xi}^j) \big).$$

Now, comparing the asymptotic representations (6.62) and (6.63) for the same function $u^\varepsilon(\mathbf{x})$ in the matching zone $\{\boldsymbol{\xi}^j : \ \varepsilon^{-1/2} d/2 \leq |\boldsymbol{\xi}^j| \leq \varepsilon^{-1/2} d\}$, or, which is the same, for $\varepsilon^{1/2} d/2 \leq |\mathbf{x} - P^j| \leq \varepsilon^{1/2} d$, we find that

$$\varepsilon v_1(\mathbf{x}) + \varepsilon^2 v_2(\mathbf{x}) - \big\{ w_0^j(\boldsymbol{\xi}^j) + \varepsilon w_1^j(\boldsymbol{\xi}^j) \big\} = O(\varepsilon^{3/2}). \tag{6.64}$$

Formula (6.64) justifies the formal two-term asymptotics constructed.

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