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Cracks propagation in ceramic materials: a singular integral formulation

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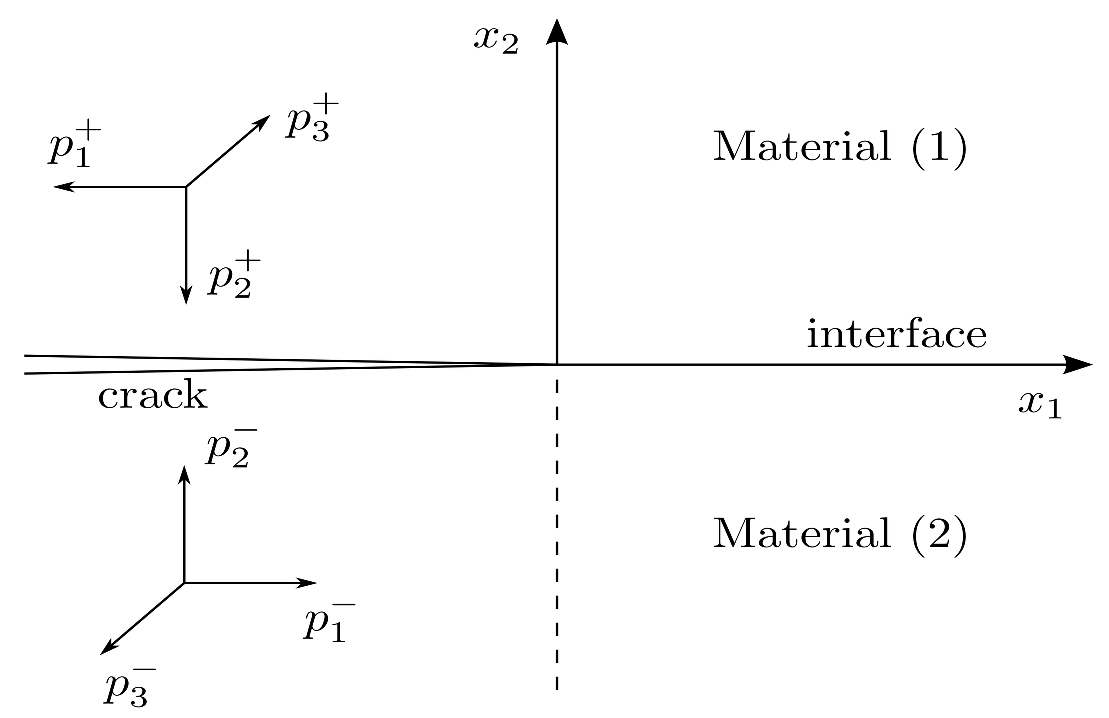
Interfacial cracks in anisotropic ceramic bimaterials



- Quasi-static semi-infinite plane crack between dissimilar anisotropic ceramic materials;
- General non-symmetric loading acting on the crack faces;

Symmetric and skew-symmetric weight functions

- $\mathbf{U}$ singular solutions of the homogeneous traction-free problem;
- Symmetric and skew-symmetric weight functions:

$$[\mathbf{U}](x_1) = \mathbf{U}(x_1, x_2 = 0^+) - \mathbf{U}(x_1, x_2 = 0^-),$$
$$\langle \mathbf{U} \rangle (x_1) = \frac{1}{2}\left(\mathbf{U}(x_1, x_2 = 0^+) + \mathbf{U}(x_1, x_2 = 0^-)\right);$$

- General expressions in Fourier space for anisotropic materials:

$$[\hat{\mathbf{U}}]^+(\xi) = -\frac{1}{|\xi|}\left\{\mathrm{Re}\mathbf{H} - i\,\mathrm{sign}(\xi)\,\mathrm{Im}\mathbf{H}\right\}\hat{\mathbf{\Sigma}}^-(\xi),$$
$$\langle \hat{\mathbf{U}} \rangle(\xi) = -\frac{1}{2|\xi|}\left\{\mathrm{Re}\mathbf{W} - i\,\mathrm{sign}(\xi)\,\mathrm{Im}\mathbf{W}\right\}\hat{\mathbf{\Sigma}}^-(\xi), \quad \xi \in \mathrm{R};$$

  - $\mathbf{H} = \mathbf{Y}^{(1)} + \bar{\mathbf{Y}}^{(2)}$ and $\mathbf{W} = \mathbf{Y}^{(1)} - \bar{\mathbf{Y}}^{(2)}$;
  - $\mathbf{Y}$ surface admittance tensor;
  - Monoclinic and orthotropic ceramics $\Rightarrow$ decoupling plane and antiplane strain;

Integral identities: antiplane strain

- Betti identity for antiplane strain:

$$\hat{\mathcal{T}}_3^+(\xi) - B[\hat{u}_3]^-(\xi) = -\langle \hat{p}_3 \rangle(\xi) - A[\hat{p}_3](\xi), \quad \xi \in \mathrm{R},$$

Where: $A = [\hat{U}_3]^{-1}\langle \hat{U}_3 \rangle$; $B = [\hat{U}_3]^{-1}\hat{\Sigma}_3$;

- Inverting the Fourier transform, we get the integral identities:

$$\langle p_3 \rangle + \frac{\eta}{2}[p_3] = -\frac{1}{H_{33}}\mathcal{S}^{(s)}\frac{\partial [u_3]^{(-)}}{\partial x_1}, \quad x_1 < 0,$$
$$\mathcal{T}_3^{(+)} = -\frac{1}{H_{33}}\mathcal{S}^{(c)}\frac{\partial [u_3]^{(-)}}{\partial x_1}, \quad x_1 > 0;$$

  - H_{33} bimaterial matrix $\boldsymbol{H}$ element, η Dundurs parameter;
  - $\mathcal{S}^{(s)} = \mathcal{P}_-\mathcal{S}\mathcal{P}_-$ and $\mathcal{S}^{(c)} = \mathcal{P}_+\mathcal{S}\mathcal{P}_-$;
  - $\mathcal{S}$ is a singular integral operator:
  $$\psi = \mathcal{S}\varphi = \frac{1}{\pi x_1} * \psi(x_1) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(\zeta)}{x_1 - \zeta}d\zeta;$$
  - $\mathcal{P}_\pm$ are orthogonal projectors ($\mathcal{P}_+ + \mathcal{P}_- = \mathcal{I}$);

Integral identities: plane strain

- Betti identity for plane strain:

$$\hat{\boldsymbol{\mathcal{T}}}^+(\xi) - \mathbf{B}[\hat{\mathbf{u}}]^-(\xi) = -\langle \hat{\mathbf{p}} \rangle(\xi) - \mathbf{A}[\hat{\mathbf{p}}](\xi), \quad \xi \in \mathrm{R},$$

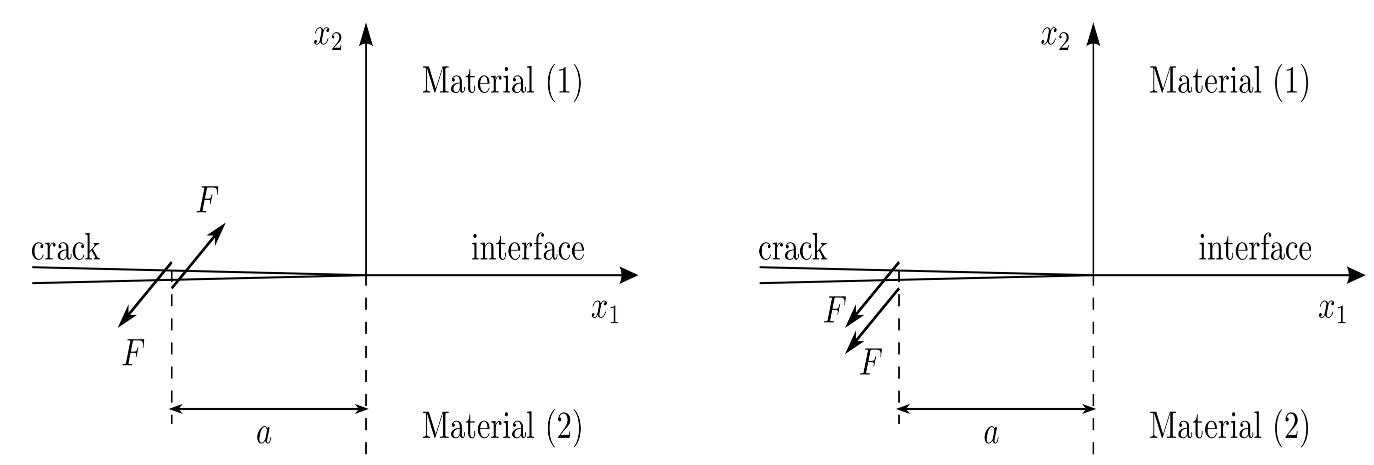
Where: $\boldsymbol{A} = \boldsymbol{R}^{-1}[\hat{\boldsymbol{U}}]^{-T}\langle \hat{\boldsymbol{U}}^T \rangle \boldsymbol{R}$; $\boldsymbol{B} = \boldsymbol{R}^{-1}[\hat{\boldsymbol{U}}]^{-T}\hat{\boldsymbol{\Sigma}}^T\boldsymbol{R}$;

- Inverting the Fourier transform, we get the integral identities:

$$\langle \boldsymbol{p} \rangle + \boldsymbol{\mathcal{A}}^{(s)}[\boldsymbol{p}] = \boldsymbol{\mathcal{B}}^{(s)}\frac{\partial [\boldsymbol{u}]^{(-)}}{\partial x_1}, \quad x_1 < 0,$$
$$\boldsymbol{\mathcal{T}}^{(+)} + \boldsymbol{\mathcal{A}}^{(c)}[\boldsymbol{p}] = \boldsymbol{\mathcal{B}}^{(c)}\frac{\partial [\boldsymbol{u}]^{(-)}}{\partial x_1}, \quad x_1 > 0;$$

  - $\boldsymbol{\mathcal{A}}^{(s)}, \boldsymbol{\mathcal{B}}^{(s)}$ matrix singular integral operators depending by $\mathcal{S}^{(s)}$;
  - $\boldsymbol{\mathcal{A}}^{(c)}, \boldsymbol{\mathcal{B}}^{(c)}$ matrix compact integral operators depending by $\mathcal{S}^{(c)}$;

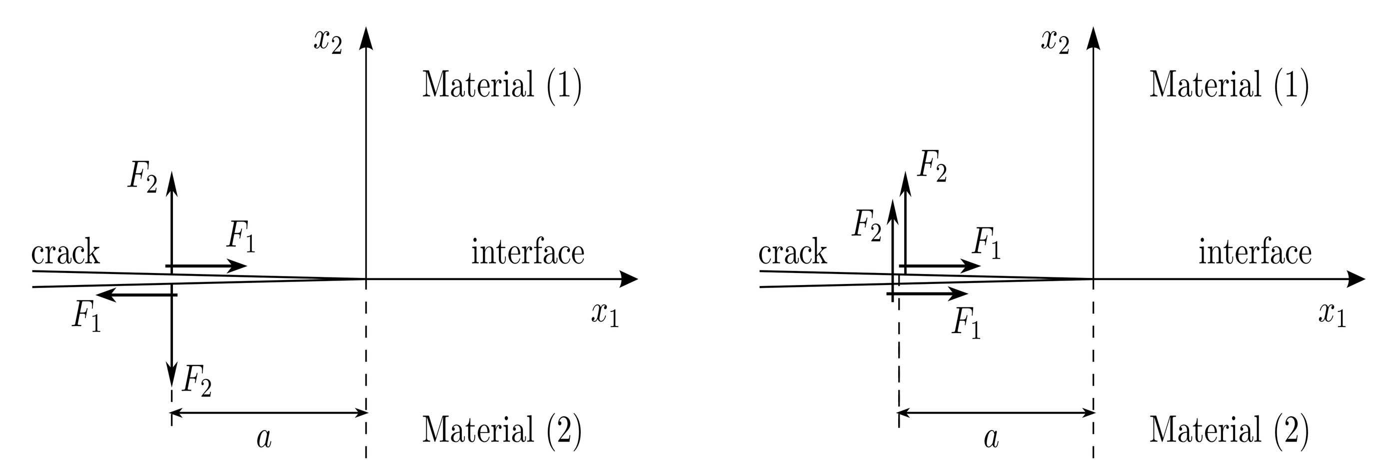
Applications: Mode III cracks



(A) Mode III: symmetrical forces (B) Mode III: skew-symmetrical forces

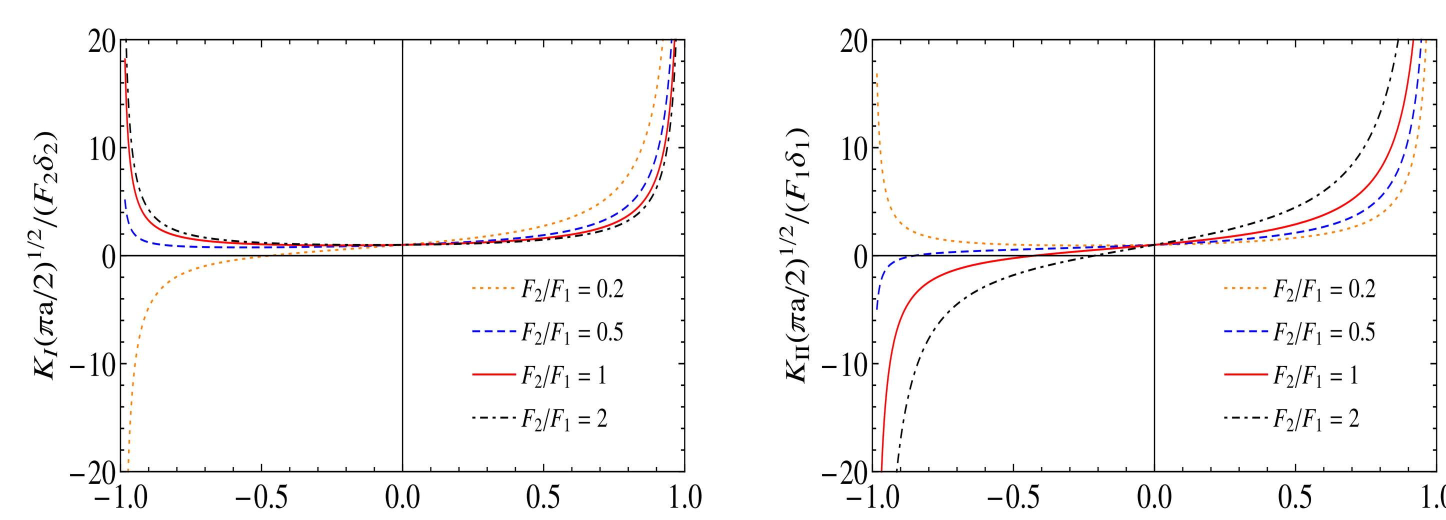
- Symmetrical forces:
  - Traction ahead of the crack tip:
  $$\mathcal{T}_3^{(+)}(x_1) = \frac{F}{\pi}\sqrt{\frac{a}{x_1}}\frac{1}{x_1 + a};$$
  - Stress intensity factors:
  $$K_{III} = \lim_{x_1 \to 0}\sqrt{2\pi x_1}\,\mathcal{T}_3^{(+)}(x_1) = \sqrt{\frac{2}{\pi a}}F;$$
- Skew-symmetrical forces:
  - Traction ahead of the crack tip:
  $$\mathcal{T}_3^{(+)}(x_1) = \eta\frac{F}{\pi}\sqrt{\frac{a}{x_1}}\frac{1}{x_1 + a};$$
  - Stress intensity factors:
  $$K_{III} = \lim_{x_1 \to 0}\sqrt{2\pi x_1}\,\mathcal{T}_3^{(+)}(x_1) = \eta\sqrt{\frac{2}{\pi a}}F;$$
  - For homogeneous material $\eta = 0$, no contribution to Mode III;

Applications: Mode I and II cracks



(A) Mode I and II: symmetrical forces (B) Mode I and II: skew-symmetrical forces

- Symmetrical forces:
  - Stress intensity factors:
  $$K_I = \lim_{x_1 \to 0}\sqrt{2\pi x_1}\,\mathcal{T}_2^{(+)}(x_1) = \sqrt{\frac{2}{\pi a}}F_2, \quad K_{II} = \lim_{x_1 \to 0}\sqrt{2\pi x_1}\,\mathcal{T}_1^{(+)}(x_1) = \sqrt{\frac{2}{\pi a}}F_1;$$
- Skew-symmetrical forces:
  - Stress intensity factors:



  - $\alpha, \delta_1, \delta_2$ Dundurs parameters $\Rightarrow$ interface oscillations;
  - For homogeneous material $\Rightarrow \delta_1, \delta_2 = 0 \Rightarrow K_I, K_{II} = 0$;

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Acknowledgements

LM gratefully thanks financial support from the Italian Ministry of Education, University and Research in the framework of the FIRB project 2010 "Structural mechanics models for renewable energy applications", AP and GM acknowledge the financial support provided by FP7 PEOPLE - IAPP - 2011-286110 "INTERCER2".

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