Nonlinear transmission conditions for thin curvilinear low-conductive interphases in composite ceramics

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Introduction

Composite materials with thin interphases are widely used in modern industry, as their structure enhances thermal and mechanical properties of the material. However, the common approach using finite elements method for modelling of such materials can be both challenging and inaccurate, possibly even leading to numerical instability, which is, clearly, not acceptable.
To tackle this problem, the interphase is replaced in mathematical and numerical models with an infinitesimal (zero thickness) object modelled by transmission conditions [1–3].

Problem formulation

Modelling problem: a domain containing thin curvilinear interphase with smooth boundaries, where the heat transfer equation is satisfied

$$\nabla \cdot (k \nabla T) + Q = c\rho \frac{\partial T}{\partial t}, \tag{1}$$

where $T(x, y)$ is the unknown temperature, $Q(T, x, y)$ the thermal source, $k(T, x, y)$ the thermal conductivity, c heat capacity and ρ the density of the material.

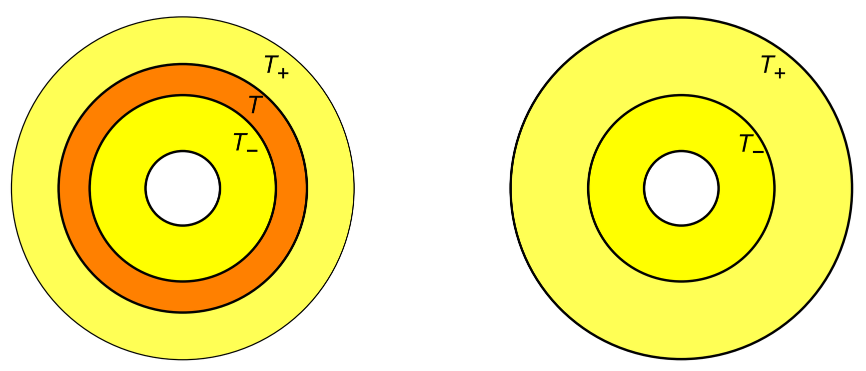


Figure 1: Domains with circular thin interphase (left) and zero thickness interface (right)

Assumptions: a) Q does not change its sign within the interphase, b) the boundaries of the interphase is comparatively small and does not change its sign, c) transmission conditions along the boundaries $\Gamma_\pm$ are:

$$[T]\,|_{\Gamma_\pm} = 0, \quad [\mathbf{nq}]\,|_{\Gamma_\pm} = 0, \tag{2}$$

where $\mathbf{n}$ is the normal vector to the surface and $\mathbf{q}$ the heat flux, defined by Fourier's law:

$$\mathbf{q} = -k(T)\nabla T. \tag{3}$$

Transition to polar coordinates

Due to the shape of the interphase it makes sense to switch to polar coordinates. Then equation 1 and the transmission conditions are:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + Q = c\rho\frac{\partial T}{\partial t}, \tag{4}$$

$$T_\pm - T(r_\pm, \phi, t) = 0, \tag{5}$$

$$q_\pm + n_r^\pm k \frac{\partial}{\partial r} T(r_\pm, \phi, t) + n_\phi^\pm k \frac{1}{r_\pm}\frac{\partial}{\partial \phi} T(r_\pm, \phi, t) = 0 \tag{6}$$

where

$$\mathbf{n}_\pm = [n_r^\pm, n_\phi^\pm] = \frac{\left[r_\pm(\phi), r'_\pm(\phi)\right]}{\sqrt{r_\pm^2(\phi) + (r'_\pm(\phi))^2}}. \tag{7}$$

Rescaling the interphase

Small parameters in the model: a) thickness of the interphase $h << 1$, b) low heat conductivity $k << k_\pm$
Asymptotic procedure:

$$h(\phi) = \varepsilon \widetilde{h}(\phi), \quad k(T, r, \phi) = \varepsilon \widetilde{k}(T, \xi, \phi),$$
$$\xi = \frac{r - r_0(\phi)}{\varepsilon \widetilde{h}(\phi)}, \quad Q(T, r, \phi) = \frac{1}{\varepsilon}\widetilde{Q}(T, \xi, \phi).$$

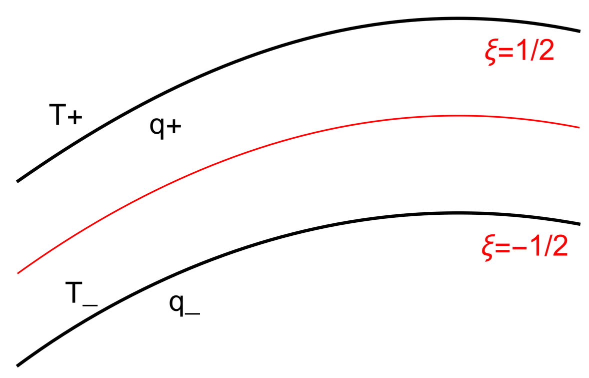


Figure 2: The rescaled interphase

After inserting into (4), the main terms of the expansion give:

$$\frac{1}{\widetilde{h}^2}\frac{\partial}{\partial \xi}\left(\widetilde{k}\frac{\partial \widetilde{T}_0}{\partial \xi}\right) + \widetilde{Q} = 0. \tag{8}$$

General algorithm

1. Integrate (8):

$$f(T, \xi, C_0, C_1) = 0. \tag{9}$$

2. Differentiate the integral:

$$\frac{-n_\xi^0(\phi)\widetilde{k}}{\widetilde{h}}\frac{\partial}{\partial \xi} f(T, \xi, C_0, C_1) = \widetilde{q}_\xi \frac{\partial}{\partial T} f(T, \xi, C_0, C_1) \tag{10}$$

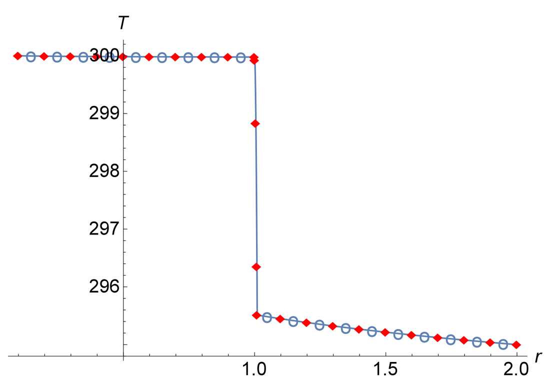
3. Insert the values at the boundaries of the interphase (see Fig.2) into (9) and (10) to obtain two conditions:

$$F_1(T_+, T_-, q_+, q_-) = 0, \tag{11}$$
$$F_2(T_+, T_-, q_+, q_-) = 0.$$

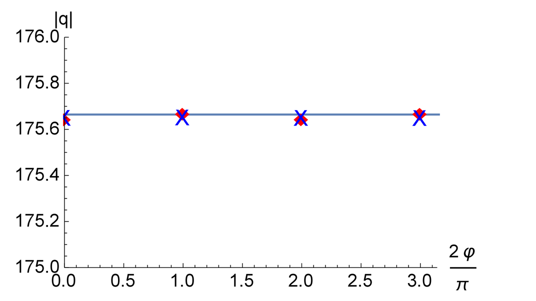
Transmission conditions were evaluated for 3 special cases of Q and k. The results presented here were derived for $\widetilde{Q} = \widetilde{Q}(T, \phi, t)$, $\widetilde{k} = \widetilde{k}(T, \phi, t)$.

Results

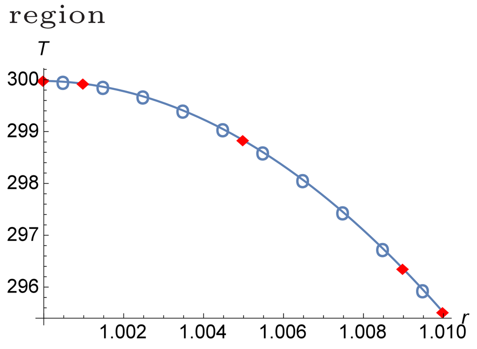
Results for circular interphase a) analytic solutions, b) solutions to the problem with transmission conditions (circular markers) and c) values from COMSOL; close-ups for the interphase; the heat flux along the boundaries: exact (solid line) and FEM-modelled (markers).



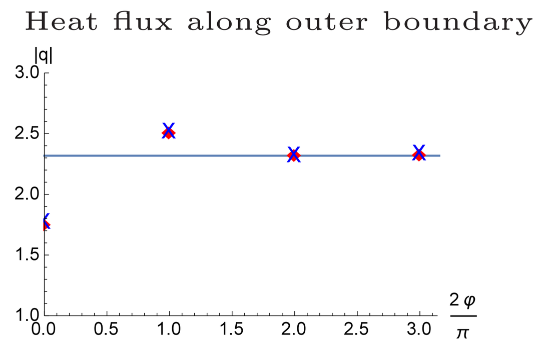
Temperature within the whole region



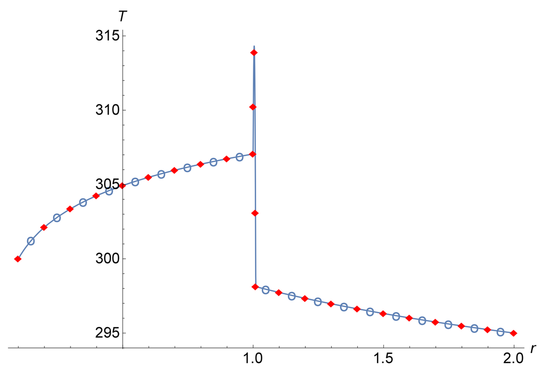
Heat flux along outer boundary



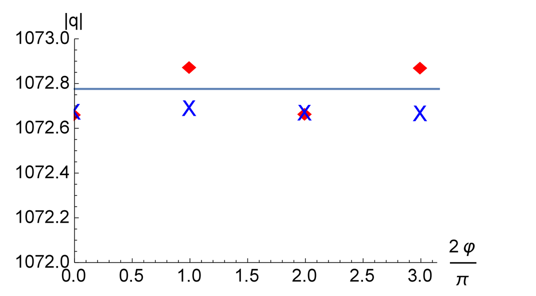
Monotonic temperature within the interphase



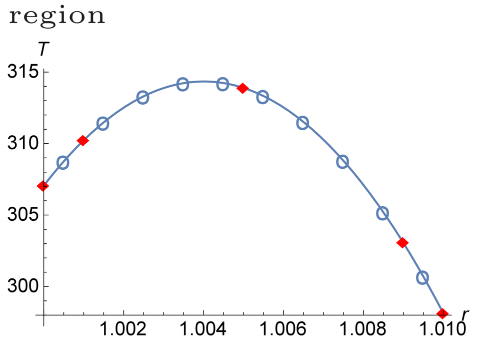
Heat flux along inner boundary



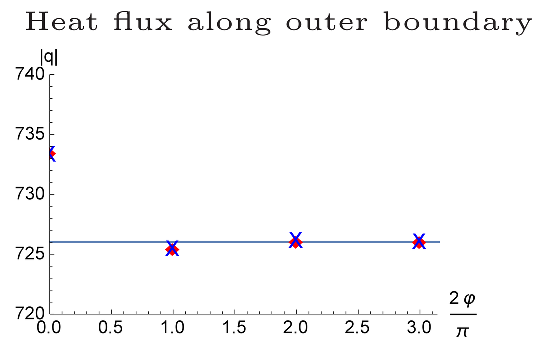
Temperature within the whole region



Heat flux along outer boundary

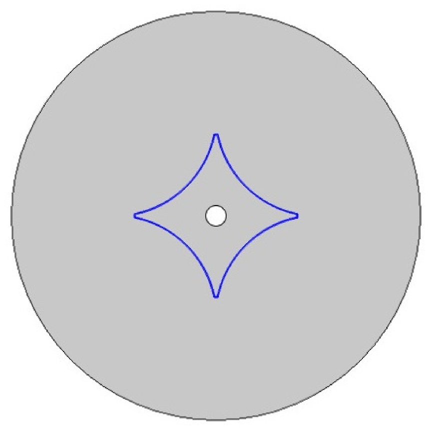


Non-monotonic temperature within the interphase

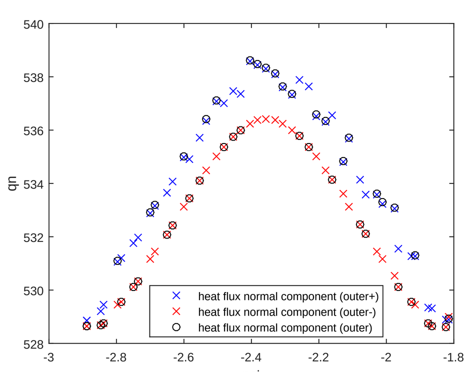


Heat flux along inner boundary

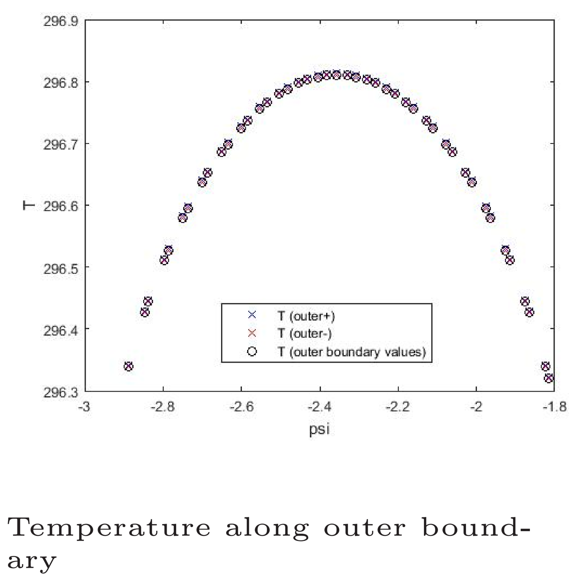
Results for star-shaped interphase COMSOL gives too big a discrepancy (for heat flux). The graphs show: the geometry, COMSOL values for temperature and for heat flux.



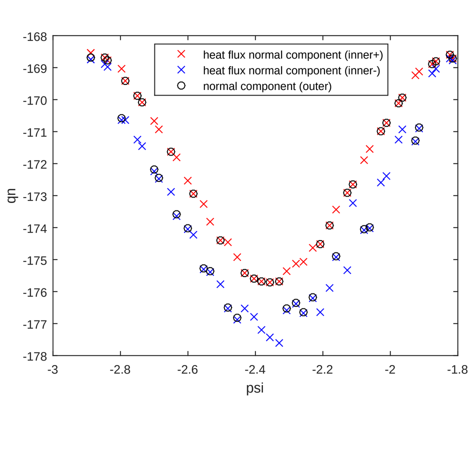
Star-shaped interphase



Heat flux along outer boundary



Temperature along outer boundary



Heat flux along inner boundary

Conclusions

Numerical verifications prove that the derived conditions work and give a small error ($O(\epsilon^2)$). At the same time, COMSOL provides less accurate results for the same structure with a complex geometry (especially for heat flux) .

Acknowledgements

DA and WM thank FP7 Marie Curie projects INTERCER2 and PARM2 respectively.
The authors acknowledge Prof. G.Mishuris for useful discussions.

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