ANALYTICAL TREATMENT OF HEAT CONDUCTION IN 2D NONLINEAR DOUBLY PERIODIC COMPOSITES

Sergei Rogosin*,†, Ekaterina Pesetskaya, Gennady Mishuris

* Belarusian State University,
Nezavisimosti ave 4, 220030 Minsk, Belarus

Fax: +375 17 209 53 32; Emails: rogosin@bsu.by; rogosinsv@gmail.com

† Speaker

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The aim of the report:

♡ to construct an analytic solution to the heat conduction problem for 2D nonlinear composite materials with doubly periodic structure.

The plan:

♠ historical remarks;

♠ mathematical model;

♠ from quasi-linear to linear equation, reformulation of boundary value problem;

♠ introduction of complex potentials, complex form of the problem;

♠ description of an algorithm;

♠ discussion.

Historical remarks.

$\diamondsuit$ Reley (1892) & Maxwel-Garnett (1904) – effective properties of inhomogeneous media

$\diamondsuit$ Berdichevskii (1975) & Bakhvalov–Panasenko (1984) & Grigolyuk–Filshtinskii (1992) – periodic composites

$\diamondsuit$ Golden–Papanicolaou (1993) & Zhikov (1994) – homogenization of a random media

$\diamondsuit$ Nguetseng (1989) & Allaire (1992) & Zhikov–Smyshlyaev–Cherednichenko (2000-2006) – two-scales homogenization

$\diamondsuit$ Nicorovici–McPhedran (1996) – Reley method

$\diamondsuit$ Mityushev (1998) – boundary value problems for analytic functions

$\diamondsuit$ Linkov (1999-2009) – boundary integral equations

$\diamondsuit$ Telega–Tokarzewski–Galka–Andrianov (2001) – nonlinear composite materials

$\diamondsuit$ Maz'ya–Movchan (2009-2011) – asymptotic expansion of Green's function

Mathematical model.

$T = T(x,y) \in \mathcal{C}^2(D_{matrix} \cup D_{inc}) \cap \mathcal{C}^1(cl(D_{matrix}) \cup cl(D_{inc}))$:

$$\nabla(\lambda(T)\nabla T) = 0,\ z \in D_{matrix}, \tag{1}$$

$$\nabla(\lambda_k(T)\nabla T) = 0,\ z \in \bigcup_{m_1,m_2} (D_k + m_1 + \imath m_2); \tag{2}$$

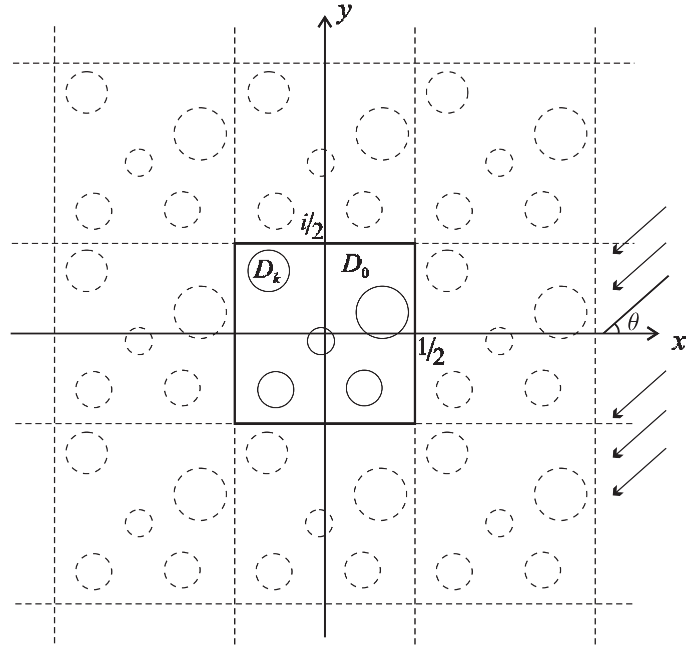
$$T(t) = T_k(t), \quad t \in \bigcup_{m_1,m_2\in\mathbb{Z}} (\partial D_k + m_1 + \imath m_2), \tag{3}$$

$$\lambda(T(t))\frac{\partial T(t)}{\partial n} = \lambda_k(T_k(t))\frac{\partial T_k(t)}{\partial n}, \quad t \in \bigcup_{m_1,m_2\in\mathbb{Z}} (\partial D_k + m_1 + \imath m_2); \tag{4}$$

$$\int_{-1/2}^{1/2} \lambda\Big(T\Big(x,\tau\Big)\Big)T_y\Big(x,\tau\Big)d\tau = -A\sin\theta, \tag{5}$$

$$\int_{-1/2}^{1/2} \lambda\Big(T\Big(\tau,y\Big)\Big)T_x\Big(\tau,y\Big)d\tau = -A\cos\theta. \tag{6}$$

Statement of the problem. Geometry.



"Central" cell $Q_{(0,0)}$ of a two-periodic composite.

$$Q_{(m_1,m_2)} = Q_{(0,0)} + m_1 + \imath m_2 := \left\{ z \in \mathbb{C} : z - m_1 - \imath m_2 \in Q_{(0,0)} \right\}.$$

$$D_k := \{ z \in \mathbb{C} : |z - a_k| < r_k \},\ D_0 := Q_{(0,0)} \setminus \left(\bigcup_{k=1}^{N} D_k \cup \partial D_k \right).$$

Statement of the problem. Physical assumptions.

Steady state potential heat conduction in 2D doubly periodic composites; unknown - temperature $T(x,y)$ and/or heat flux $\mathbf{q}(x,y)$.

Assumptions:

- conductivity of matrix $\lambda(T)$ and inclusions $\lambda_k(T)$ are bounded continuously differentiable functions on $\mathbb{R}$:

$$0 < \lambda^- \leq \lambda(t) \leq \lambda^+ < +\infty,$$

$$0 < \lambda_k^- \leq \lambda_k(t) \leq \lambda_k^+ < +\infty, \quad k = 1, \ldots, N;$$

$$|\lambda'(t)| \leq \mu^+ < +\infty, \quad |\lambda_k'(t)| \leq \mu_k^+ < +\infty, \quad k = 1, \ldots, N;$$

- ideal contact condition;

- steady external flow at infinity.

Reduction to linear equation.

$$f(T) = \int_0^T \lambda(\xi)\, d\xi, \quad f_k(T) = \int_0^T \lambda_k(\xi)\, d\xi, \quad k = 1, \ldots, N; \tag{7}$$

$$u(z) = f(T(z)), \quad u_k(z) = f_k(T(z)), \quad k = 1, \ldots, N. \tag{8}$$

$$\Delta u(z) = 0, \quad z \in D_{matrix}, \tag{9}$$

$$\Delta u_k(z) = 0, \quad z \in \bigcup (D_k + m_1 + \imath m_2). \tag{10}$$

$$u(t) = f(f_k^{-1}(u_k(t))), \tag{11}$$

$$\frac{\partial u(t)}{\partial n} = \frac{\partial u_k(t)}{\partial n}, \quad t \in \bigcup (\partial D_k + m_1 + \imath m_2). \tag{12}$$

$$\int_{-1/2}^{1/2} u_y\Big(x, \tau\Big) d\tau = -A \sin\theta, \tag{13}$$

$$\int_{-1/2}^{1/2} u_x\Big(\tau, y\Big) d\tau = -A \cos\theta. \tag{14}$$

Complex potentials.

New unknown (analytic) functions $\varphi(z)$, $\varphi_k(z)$:

$$u(z) = \text{Re}\,\{\varphi(z) + \alpha z\}, \tag{15}$$

$$\alpha = \alpha_1 + \imath\alpha_2, \quad \alpha_1 = -A\cos\theta, \quad \alpha_2 = A\sin\theta.$$

$$\varphi(z+1) - \varphi(z) = 0, \qquad \varphi(z+\imath) - \varphi(z) = 0.$$

$$v_k(z) = \text{Im}\,\{\varphi_k(z)\}, \quad z \in D_k. \tag{16}$$

Nonlinear boundary condition

$$\varphi(t) = f\left(f_k^{-1}\left(\frac{\varphi_k(t) + \overline{\varphi_k(t)}}{2}\right)\right) + \frac{\varphi_k(t) - \overline{\varphi_k(t)}}{2} - \alpha t - \frac{\varphi_k(a_k) - \overline{\varphi_k(a_k)}}{2}, \; t \in \partial D_k. \tag{17}$$

Solution algorithm.

STEP 1. Let $\tau_k = \tau_k^{(0)} = 0$, $\lambda = \lambda(0) = \lambda^{(0)}$, $\lambda_k = \lambda_k(0) = \lambda_k^{(0)}$, $\rho_k(\tau_k) = \rho_k^{(0)} = \frac{\lambda(\tau_k) - \lambda_k(\tau_k)}{\lambda(\tau_k) + \lambda_k(\tau_k)} = \frac{\lambda(0) - \lambda_k(0)}{\lambda(0) + \lambda_k(0)}$.

Let $\varphi_\alpha(z)$, $\varphi_{\alpha,k}(z)$ be a unique solution of the problem

$$\varphi_\alpha(t) = \varphi_{\alpha,k}(t) + \rho_k(\tau_k)\overline{\varphi_{\alpha,k}(t)} - \alpha t, \quad t \in \partial\, D_k, \tag{18}$$

in the space

$$\mathbf{X} = \mathcal{C}^2_{per}\,(D_0) \cap \mathcal{C}^1 \left(D_0 \bigcup_{k=1}^{N} \partial\, D_k \right) \times \mathcal{C}^2\,(D_1) \cap \mathcal{C}^1\,(cl\, D_1) \times \ldots \times \mathcal{C}^2\,(D_N) \cap \mathcal{C}^1\,(cl\, D_N)$$

$$\widetilde{\varphi}(z) = \varphi(z) - \varphi_\alpha(z), \quad z \in D, \tag{19}$$

$$\widetilde{\varphi}_k(z) = \frac{\lambda(\tau_k) + \lambda_k(\tau_k)}{2\lambda_k(\tau_k)} \cdot \left[\varphi_k(z) - \varphi_k(a_k) - \frac{2\lambda_k(\tau_k)}{\lambda(\tau_k) + \lambda_k(\tau_k)} \varphi_{\alpha,k}(z) \right], \quad z \in D_k, \tag{20}$$

Reformulation of boundary condition (17):

$$\widetilde{\varphi}(t) = \widetilde{\varphi}_k(t) + \rho_k(\tau_k)\overline{\widetilde{\varphi}_k(t)} + G_k\left(\widetilde{\varphi}_k(t), \overline{\widetilde{\varphi}_k(t)}, \lambda, \lambda_k\right)(t), \quad t \in \partial\, D_k, \quad (21)$$

where

$$G_k\left(\widetilde{\varphi}_k(t), \overline{\widetilde{\varphi}_k(t)}, \lambda, \lambda_k\right)(t) =$$

$$= f\left(f_k^{-1}\right)(f_k(T_k(t))) - \frac{\lambda(\tau_k)}{2\lambda_k(\tau_k)}\left[\varphi_k(t) - \varphi_k(a_k) + \overline{\varphi_k(t)} - \overline{\varphi_k(a_k)}\right] =$$

$$= f(T_k(t)) - \frac{\lambda(\tau_k)}{2(\lambda(\tau_k) + \lambda_k(\tau_k))}\left[\widetilde{\varphi}_k(t) + \overline{\widetilde{\varphi}_k(t)} + \varphi_{\alpha,k}(t) + \overline{\varphi_{\alpha,k}(t)}\right]. \quad (22)$$

STEP 2. Substitute

$$\widetilde{\varphi}_k(z) = \widetilde{\varphi}_k^{(0)}(z) \equiv 0, \quad \lambda = \lambda(0) = \lambda^{(0)}, \quad \lambda_k = \lambda_k(0) = \lambda_k^{(0)},$$

into nonlinear term G_k in (21) and put $\rho_k = \rho_k(0) = \rho_k^{(0)}$ in (21).

Solve an auxiliary problem in the class of periodic functions

$$G^{-,(0)}(t) - G_k^{+,(0)}(t) = G_k\left(\widetilde{\varphi}_k^{(0)}(t), \overline{\widetilde{\varphi}_k^{(0)}(t)}, \lambda^{(0)}, \lambda_k^{(0)}\right)(t), \quad t \in \partial D_k. \tag{23}$$

The following problem is equivalent to (21):

$$\widetilde{\varphi}(t) - G^{-,(0)}(t) = \widetilde{\varphi}_k(t) + \rho_k^{(0)}\overline{\widetilde{\varphi}_k(t)} - G_k^{+,(0)}(t), \quad t \in \partial D_k,\ k = 1, \ldots, N. \tag{24}$$

STEP 3. In order to solve problem (24) introduce the following function

$$\Phi(z)=\begin{cases}\widetilde{\varphi}_k(z)+\sum_{j=1}^{N}\sum_{m_1,m_2}{}^{*}\rho_j^{(0)}W_{m_1,m_2,j}\widetilde{\varphi}_j(z)-G_k^{+,(0)}(z), z\in D_k,\\ \\ \widetilde{\varphi}(z)+\sum_{j=1}^{N}\sum_{m_1,m_2}\rho_j^{(0)}W_{m_1,m_2,j}-G^{-,(0)}(z), \quad z\in D_{matrix},\end{cases} \tag{25}$$

$$W_{m_1,m_2,j}\widetilde{\varphi}_j^{(0)}(z)=\overline{\widetilde{\varphi}_j^{(0)}\left(\frac{r_k^2}{\overline{z-a_k-m_1-\imath m_2}}+a_k\right)}, \tag{26}$$

$$\sum_{j=1}^{N}\sum_{m_1,m_2}{}^{*}W_{m_1,m_2,j}:=\sum_{j\neq k}\sum_{m_1,m_2}W_{m_1,m_2,j}+\sum_{m_1,m_2}{}^{'}W_{m_1,m_2,k}. \tag{27}$$

From starting boundary conditions follows that it is identically equal to zero. System of functional equations

$$\widetilde{\varphi}_k(z) = -\sum_{j=1}^{N} \sum_{m_1,m_2}{}^{*} \rho_j^{(0)} W_{m_1,m_2,j} \widetilde{\varphi}_j(z) + G_k^{+,(0)}(z), z \in D_k. \qquad (28)$$

Solution of (28) gives the first approximation of solution $\widetilde{\varphi}^{(1)}(z), \widetilde{\varphi}_k^{(1)}(z)$. Calculate points $\tau_k^{(1)} = T_k^{(1)}(a_k)$, and values $\lambda = \lambda(\tau_k^{(1)}) = \lambda^{(1)}$, $\lambda_k = \lambda_k(\tau_k^{(1)}) = \lambda_k^{(1)}$, и $\rho_k = \rho_k(\tau_k^{(1)}) = \rho_k^{(1)}$.

STEP 4 - CYCLE.

Solve in $\mathbf{X}$ an auxiliary problem (18) with $\tau_k = \tau_k^{(1)}$ и $\lambda = \lambda(\tau_k^{(1)}) = \lambda^{(1)}$, $\lambda_k = \lambda_k(\tau_k^{(1)}) = \lambda_k^{(1)}$, and $\rho_k = \rho_k(\tau_k^{(1)}) = \rho_k^{(1)}$, and auxiliary problem (23), in which the function $\widetilde{\varphi}_k^{(0)}$ is replaced by $\widetilde{\varphi}_k^{(1)}$, and parameters $\lambda^{(0)}, \lambda_k^{(0)}$ are replaced by $\lambda^{(1)}, \lambda_k^{(1)}$.

By making corresponding substitutions we arrive at the problem, analogous to (24):

$$\widetilde{\varphi}(t) - G^{-,(1)} = \widetilde{\varphi}_k(t) + \rho_k^{(1)}\overline{\widetilde{\varphi}_k(t)} + G_k^{+,(1)}, \quad t \in \partial D_k,\ k = 1, \ldots, N. \quad (29)$$

Solving (29) we obtain the next approximation to the solution $\widetilde{\varphi}^{(2)}(z), \widetilde{\varphi}_k^{(2)}(z)$ and calculate new values of parameters.

Etc.

Thank you for your attention!