XXXX SUMMER SCHOOL „ADVANCED PROBLEMS IN MECHANICS”

ST. PETERSBURG, RUSSIA 2012

COMPLEX VARIABLE FAST MULTIPOLE METHOD FOR MODELLING HYDRAULIC FRACTURES IN INHOMOGENEOUS MEDIA

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PROBLEMS TO BE DISCUSSED

- employing special forms of the complex variable boundary integral equations (CV-BIEs)

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- employing special forms of the complex variable boundary integral equations (CV-BIEs)

- using CV-BIEs and new analytical formulae for multipole moments in frames of the Fast Multipole Method (FMM)

EMPLOYING SPECIAL FORMS OF THE COMPLEX VARIABLE BOUNDARY INTEGRAL EQUATIONS (CV-BIEs)

- types of the boundary elements
- linear transformations of coordinates for straight and circular-arc boundary elements
- approximations of higher order for the density function

Employing special forms of the complex variable boundary integral equations (CV-BIEs)

- types of the boundary elements
- linear transformations of coordinates for straight and circular-arc boundary elements
- approximations of higher order for the density function

Linkov A.M. *Boundary Integral Equations in Elasticity Theory*, Kluwer Academic Publishers, 2002;

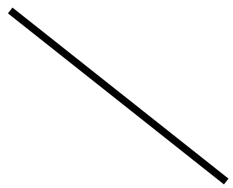
Dobroskok A.A., Linkov A.M. *Complex variable equations and numerical solution of harmonic problems for piece-wise homogeneous media*, J. Appl. Math. Mech., 2009, 73 (3), 313-325;

Linkov A.M., Szynal-Liana A. *CV circular-arc ordinary and (multi-) wedge elements for harmonic problems,Eng. Anal. Bound. Elem*, 2009, 33, 611-617; etc.

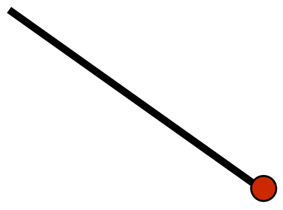
THE MAIN TYPES OF ELEMENTS USED FOR APPROXIMATION OF THE BOUNDARY OF THE REGION

- straight
  a) ordinary (non-tip)
  b) singular (tip)

a)



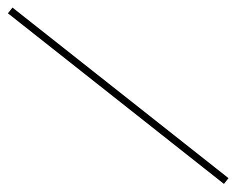
b)



THE MAIN TYPES OF ELEMENTS USED FOR APPROXIMATION OF THE BOUNDARY OF THE REGION

- straight
  - a) ordinary (non-tip)
  - b) singular (tip)
- circular-arc
  - c) ordinary (non-tip)
  - d) singular (tip)

a)



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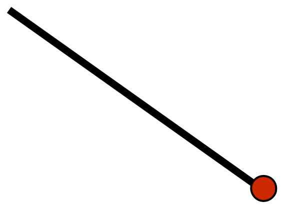

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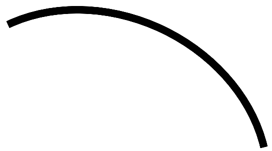

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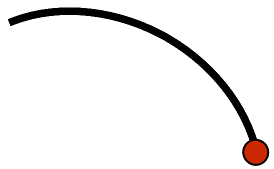
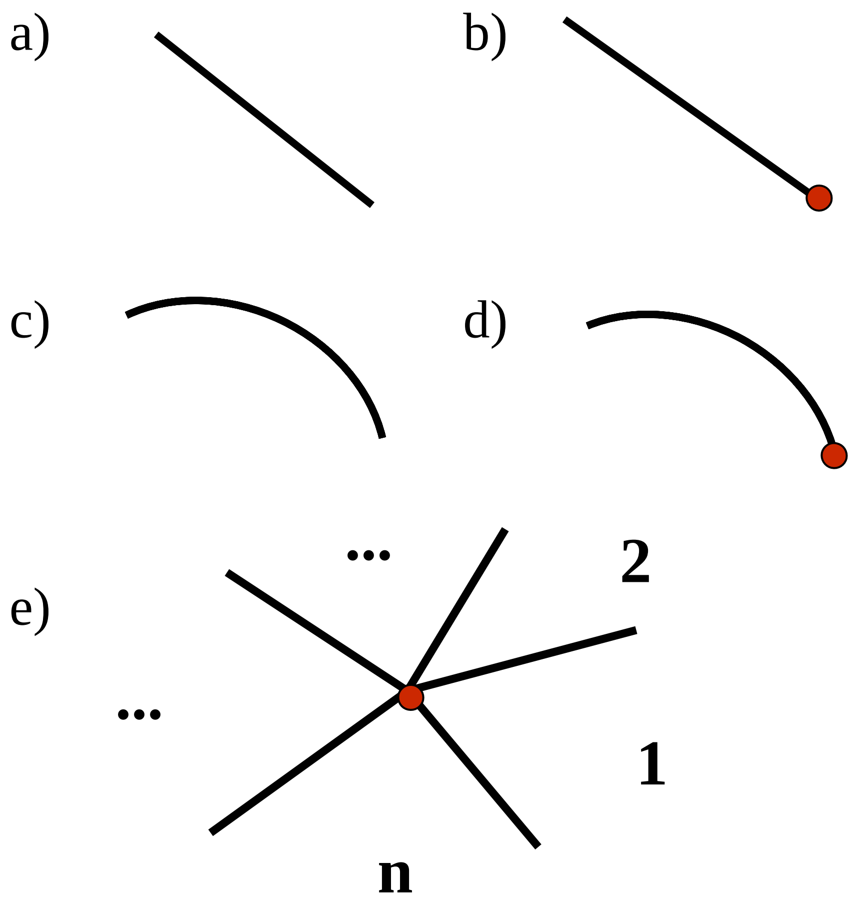


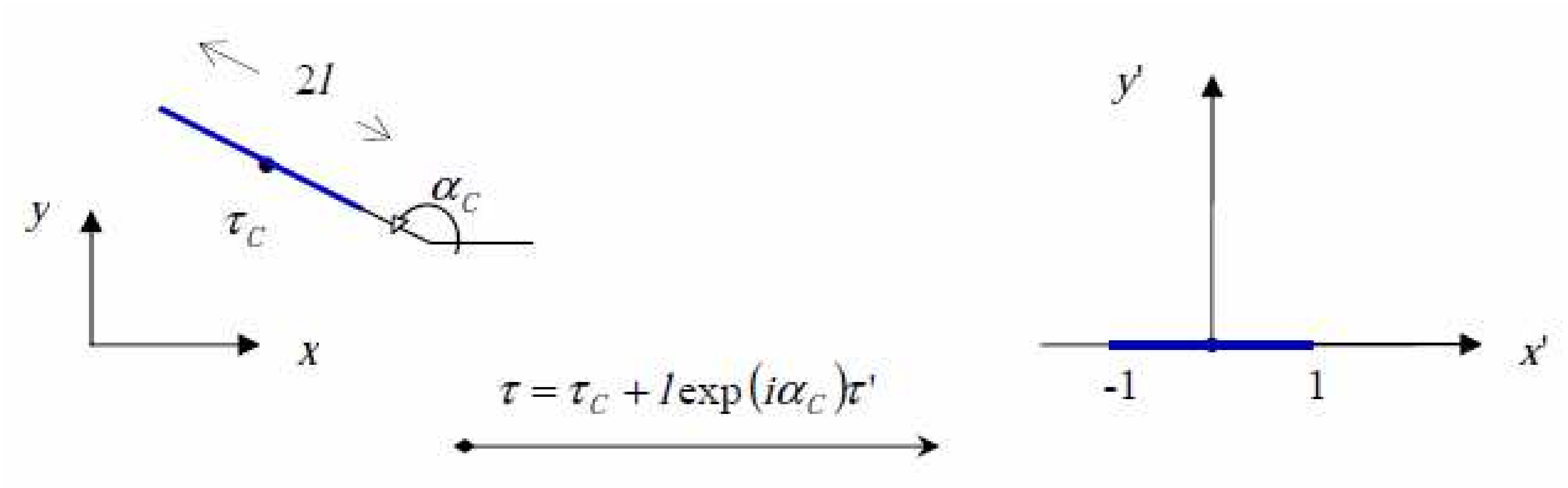
THE MAIN TYPES OF ELEMENTS USED FOR APPROXIMATION OF THE BOUNDARY OF THE REGION

- straight
  - a) ordinary (non-tip)
  - b) singular (tip)
- circular-arc
  - c) ordinary (non-tip)
  - d) singular (tip)
- special singular
  - e) multi-wedge



LINEAR TRANSFORMATION OF CV COORDINATES FROM GLOBAL TO LOCAL SYSTEM FOR STRAIGHT ELEMENT

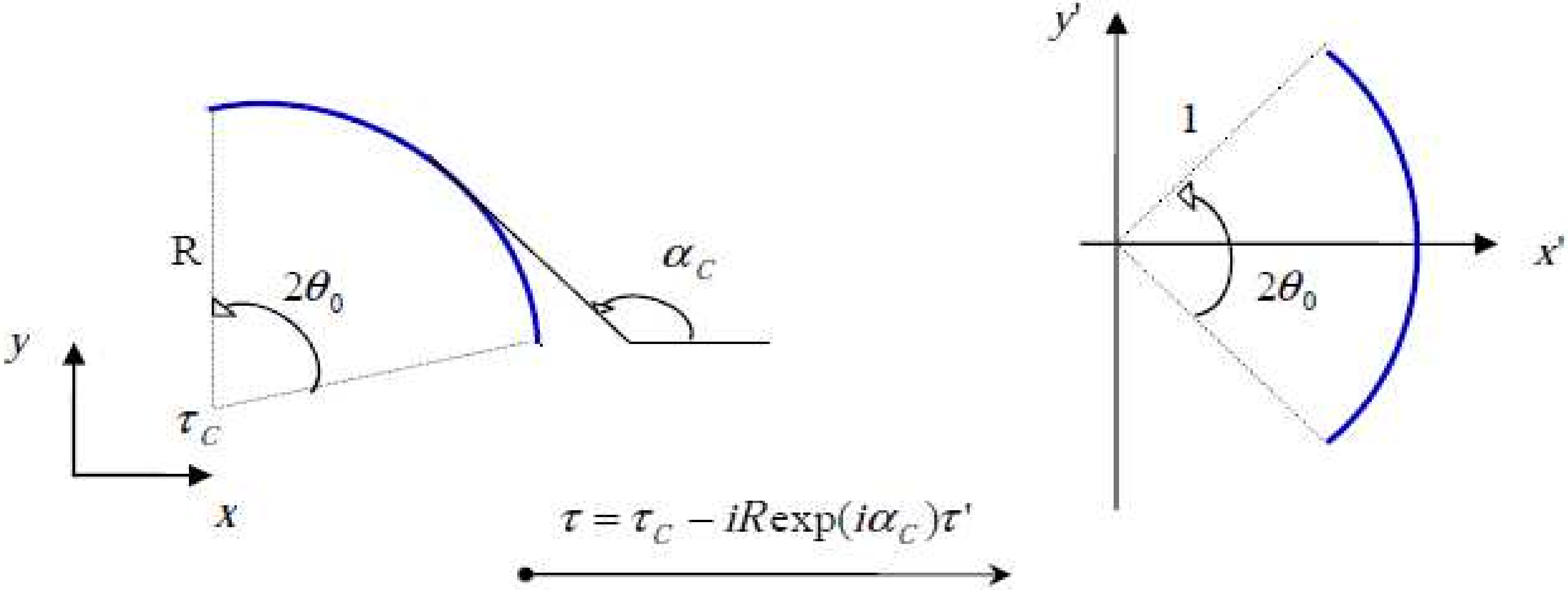
- Straight element



A straight element is transformed into standard straight element

LINEAR TRANSFORMATION OF CV COORDINATES FROM GLOBAL TO LOCAL SYSTEM FOR CIRCULAR-ARC ELEMENT

- Circular-arc element



A circular-arc element is transformed into standard circular-arc element of unit radius

SEVEN STANDARD INTEGRALS USED TO BUILD CV BOUNDARY INTEGRAL EQUATIONS (CV-BIEs)

$$\int_{L_e} \frac{f(\tau)}{\tau - z} d\tau, \quad \int_{L_e} \frac{f(\tau)}{(\tau - z)^2} d\tau, \quad \int_{L_e} f(\tau) \frac{\partial k_1}{\partial z} d\tau, \quad \int_{L_e} f(\tau) \frac{\partial k_2}{\partial z} d\tau,$$

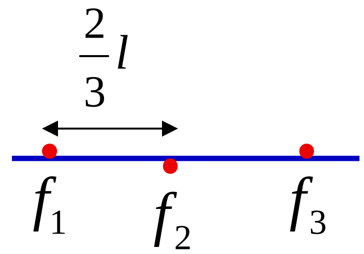
$$\int_{L_e} f(\tau)\, dk_1(\tau, z) d\tau, \quad \int_{L_e} f(\tau)\, dk_2(\tau, z) d\tau, \quad \int_{L_e} f(\tau) \ln|\tau - z| ds,$$

L_e is the boundary element, $f(\tau)$ is the density function, $z = x + iy$, τ are the CV coordinates of the field and integration points, respectively, ds is the length increment of integration path, $k_1 = Ln\left(\frac{\tau - z}{\bar{\tau} - \bar{z}}\right)$, $k_2 = \frac{\tau - z}{\bar{\tau} - \bar{z}}$.

APPROPRIATE APPROXIMATIONS OF THE DENSITY FUNCTION:

- for the standard straight element:

$$f(\tau') = \sum_{k=1}^{3} f_k \sum_{j=0}^{2} c_{kj} \tau'^{j} (1-\tau')^{\beta},$$



c_{kj}, are the Lagrange coefficients of the approximated function.

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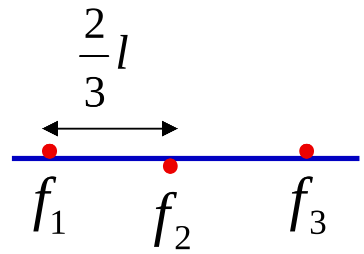
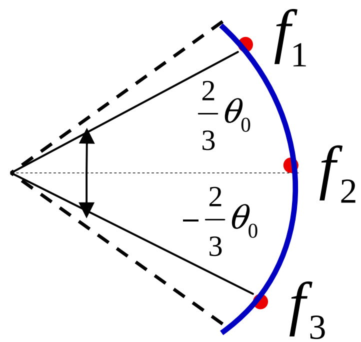


c_{kj}, are the Lagrange coefficients of the approximated function.

- for the standard circular-arc element:

$$f(\tau') = \sum_{k=1}^{3} f_k \sum_{j=-1}^{1} \tilde{c}_{kj} \tau'^{j} \operatorname{Re}\left((e^{i\theta_0} - \tau')^{\beta}\right),$$



$\tilde{c}_{kj}$, are the coefficients of the form functions at the arc of unit radius.

APPROPRIATE APPROXIMATIONS OF THE DENSITY FUNCTION: EVALUATION OF THE EXPONENT β

- for integral on ordinary (non-tip) element: $\beta = 0$
- for log-type kernel integral on singular (tip) element: $\beta = -m/n \ (m < n)$
- for singular and hypersingular integrals on singular (tip) element: $\beta = m/n > 0$

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When considering integrals on singular multi-wedge elements, the value of β is found by the methods suggested in:

Blinova V.G., Linkov A.M. „*A method to find asymptotic forms at the common apex of elastic wedges.”* J. Appl. Math. Mech., 1995, 59 (2), 187-195

Linkov A.M., Koshelev V.F. „*Muti-wedge points and multi-wedge elements in computational mechanics: evaluation of exponent and angular distribution.”* Int. J. Solids and Structures, 2006, 43, 5909-5930

USING CV-BIE AND NEW ANALYTICAL FORMULAE FOR MULTIPOLE MOMENTS IN FRAMES OF THE FAST MULTIPOLE METHOD (FMM)

- cells and quad-tree structure
- analytical reccurent formulae for log-type kernel integrals defining influence coefficients and multipole moments
- multipole and local translations

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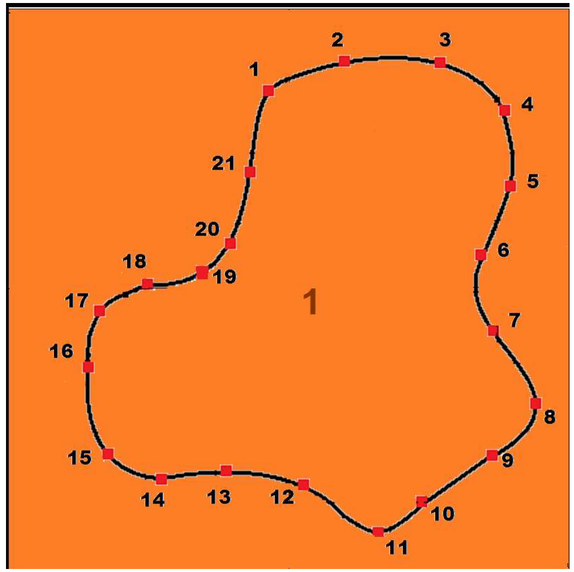
Greengard L.F., Rokhlin V. „*A fast algorithm for particle simulations*”, J Comput. Phys., 1987, 73 (2), 325-348;

Liu Y.J., Nishimura N. „*The fast multipole boundary element method for potential problems: A tutorial*”, Eng. Anal. Bound. Elem., 2006, 30, 371-381;

and many others

CELLS STRUCTURE OF THE BOUNDARY

- discretization of the boundary and preparing input data
- parent cell at level 0



- cells — **1-5 numeration of cells**
- center of the element — **1-21** numeration of points

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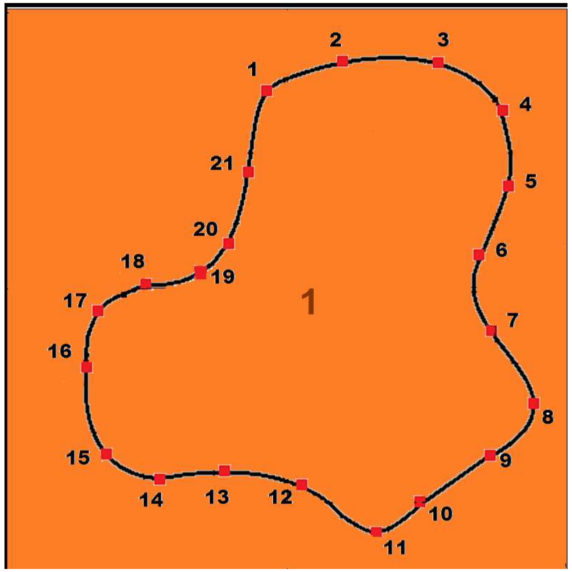
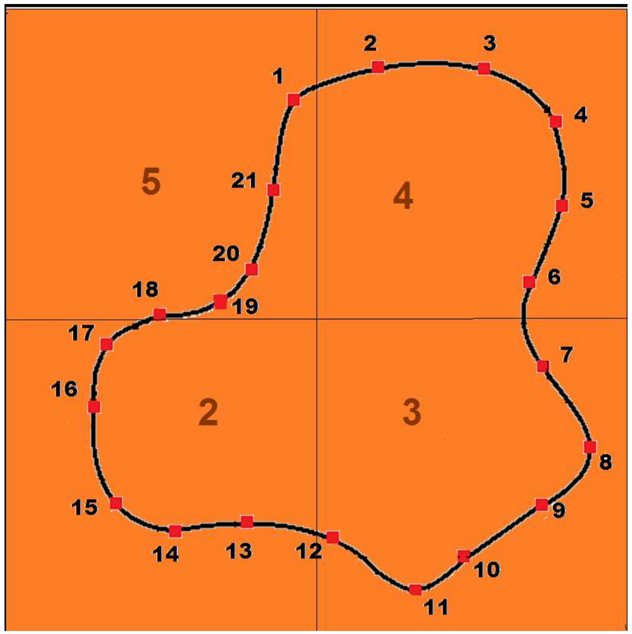


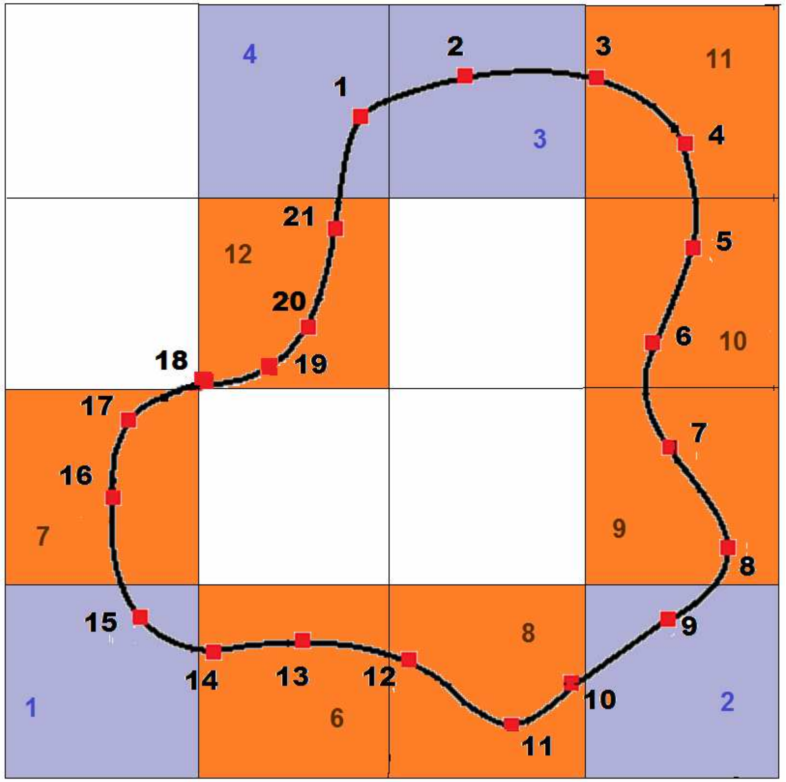
- parent cell at level 0



- dividing parent cell at level 0, child-cells at level 1

- cells — **1-5 numeration of cells**
- center of the element — **1-21** numeration of points

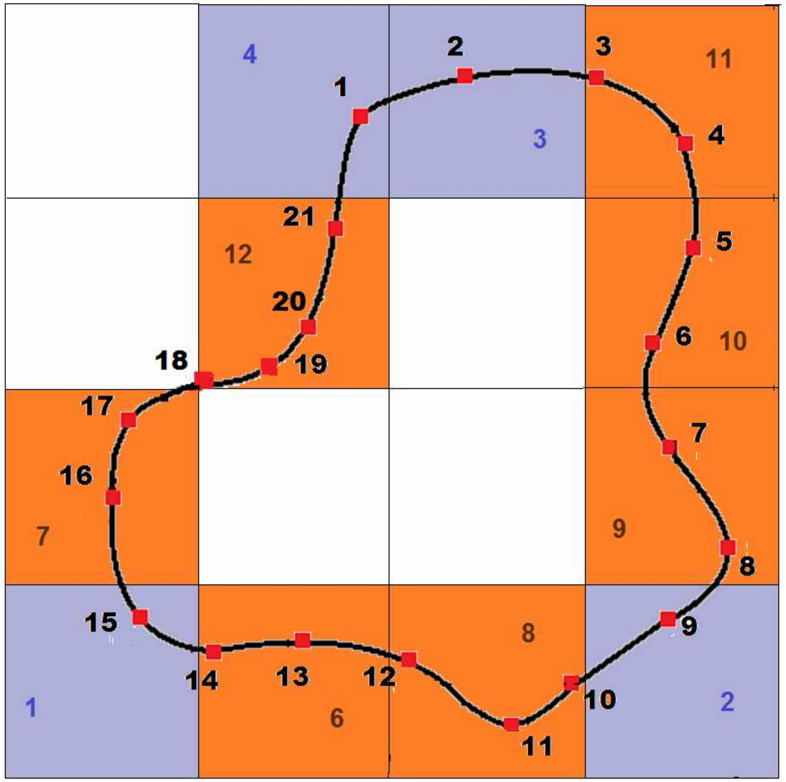
CELLS STRUCTURE OF THE BOUNDARY - NEXT STEPS OF DIVISION



- cells and leaves at the level 2

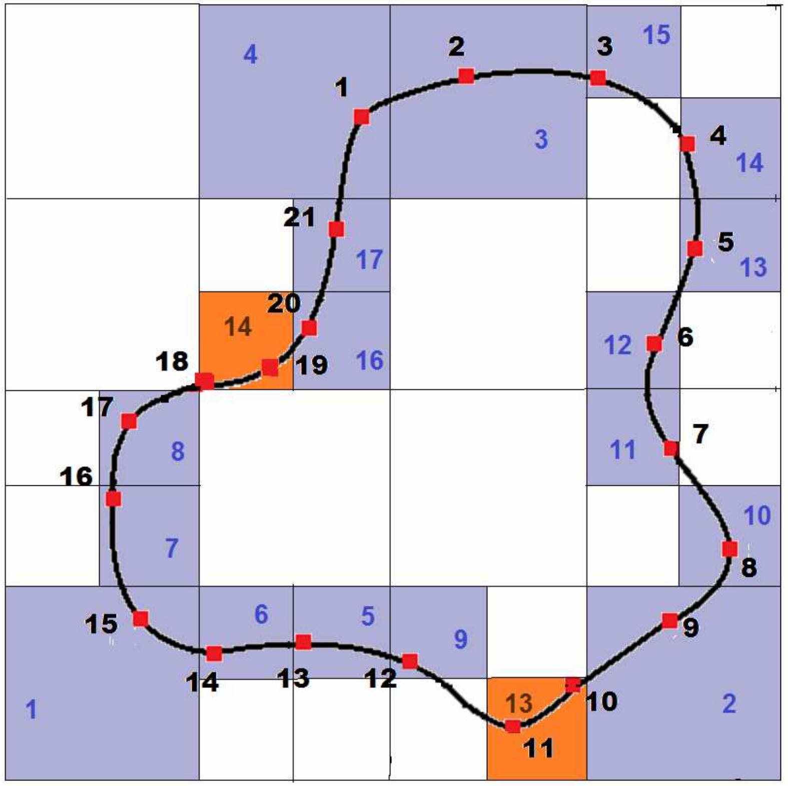
▪ cells ▪ leaves

CELLS STRUCTURE OF THE BOUNDARY - NEXT STEPS OF DIVISION



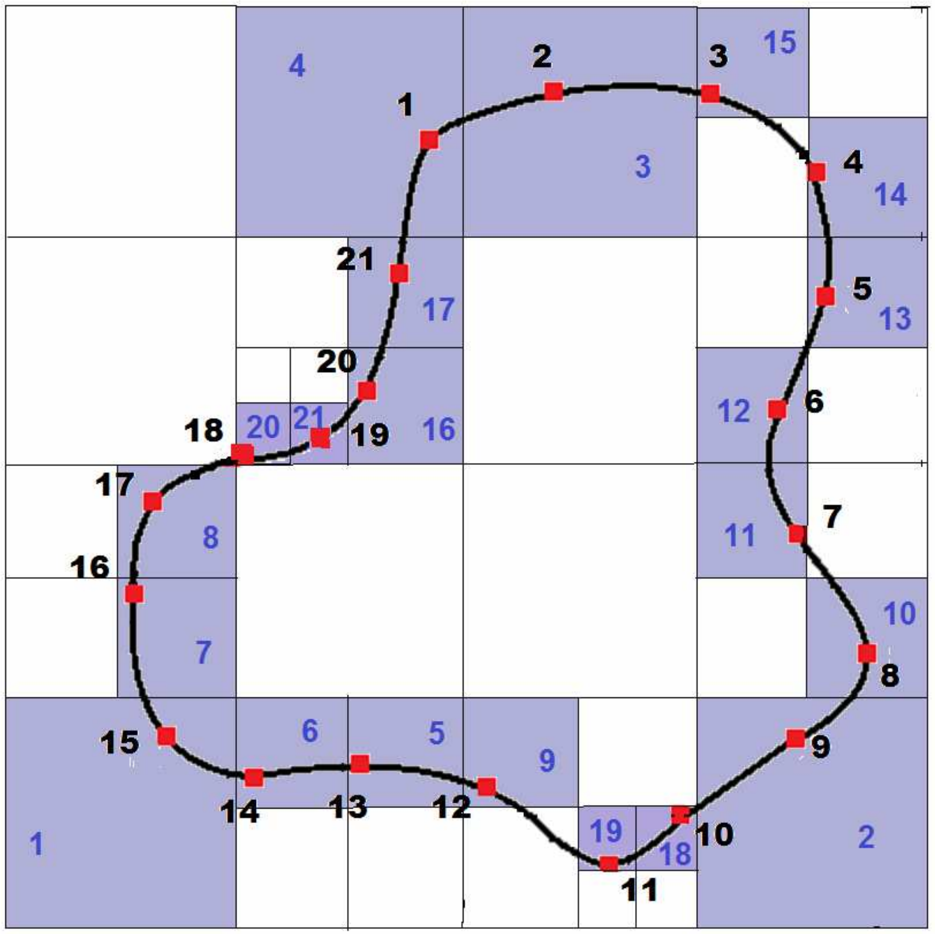
- cells and leaves at the level 2

- cells and leaves at the level 3



▪ cells ▪ leaves

FINAL SHAPE OF THE CELLS STRUCTURE OF THE BOUNDARY

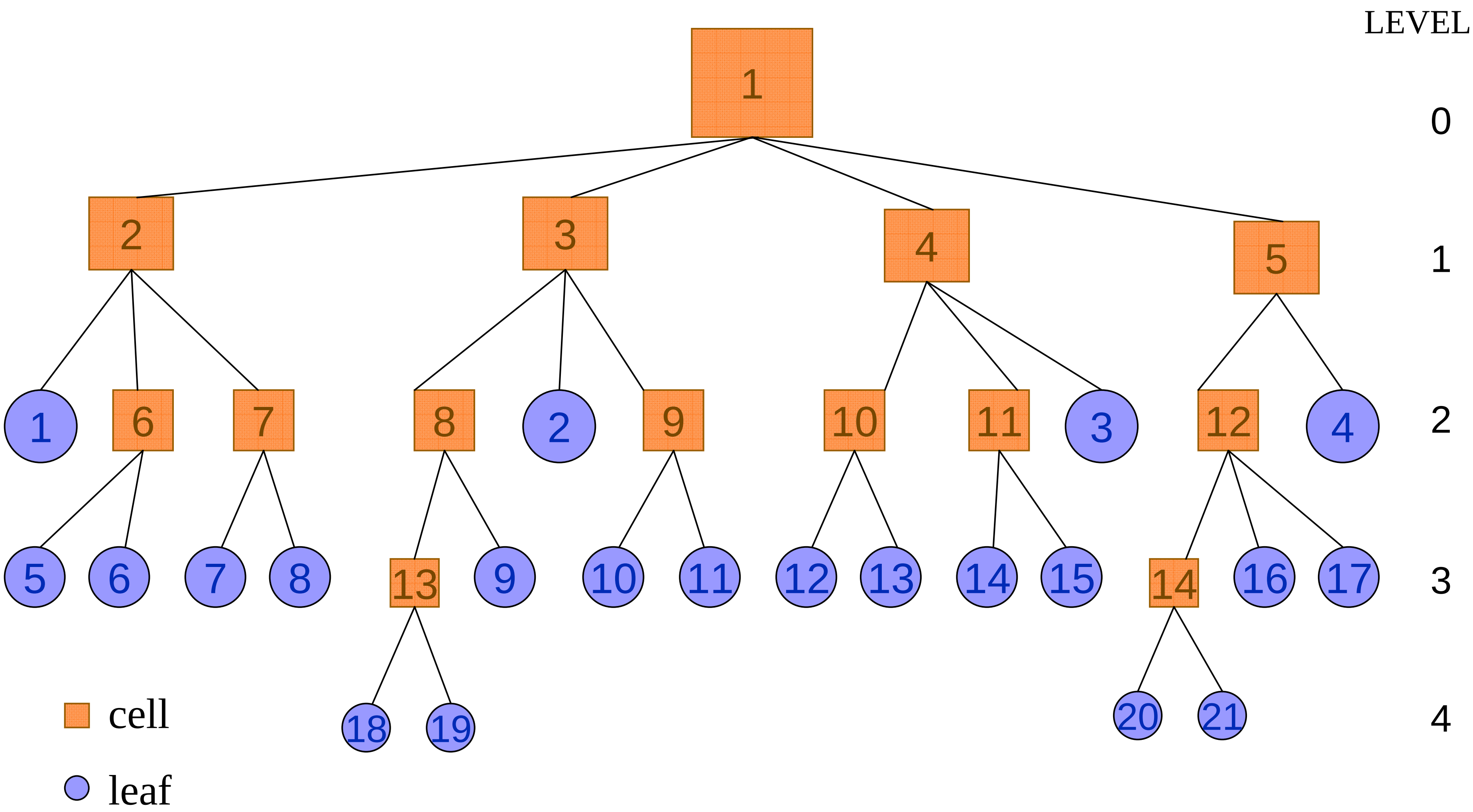


- continue dividing all non-empty cells to obtain leaves (non-empty cells containing less than prescribed number of elements)
- numeration of leaves and special renumeration of elements in each object at all levels

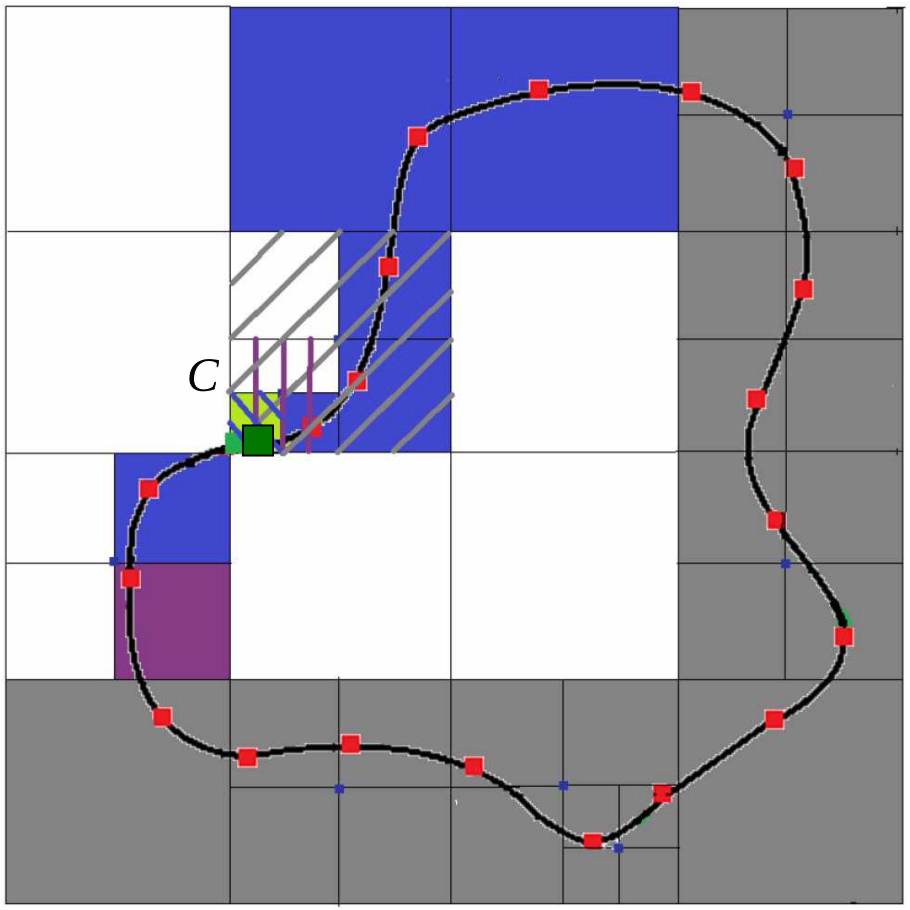
▪ leaves

1-21 numeration of leaves

THE HIERARCHICAL QUAD-TREE STRUCTURE



INTERACTIONS BETWEEN OBJECTS: ADJACENT, WELL SEPARATED AND FAR OBJECTS



- collocation point
- leaf *C*
- objects adjacent to the leaf *C*
- parent cell of leaf *C* at level 3
- objects in interaction list to the parent cell of *C* at level 3, far objects to the leaf C
- parent cell of leaf *C* at level 2
- objects in interaction list to the parent cell of *C* at level 2, far objects to the leaf C

MULTIPOLE MOMENTS - DEFINITION

Consider integral on the element L_e , contained in the leaf with the center τ_0 and the point z far from the element of integration

$$\int_{L_e} f(\tau)G(\tau,z)ds.$$

Expansion of the logarithmic potential $G(\tau,z) = -\frac{1}{2\pi}\ln|\tau - z|$ into Taylor series, yields:

$$-\frac{1}{2\pi}\int_{L_e} f(\tau)\ln|\tau - z|ds \approx \frac{1}{2\pi}\mathrm{Re}\left(\sum_{q=0}^{R_q} O_q(z-\tau_0)\int_{L_e} I_q(\tau-\tau_0)f(\tau)ds\right),$$

where the right-hand side integral is called multipole moment of order q,

$$O_0(z-\tau_0) = -Ln|z-\tau_0|, \quad O_q(z-\tau_0) = \frac{(q-1)!}{(z-\tau_0)^q} \text{ for } q \geq 1, \quad I_q(\tau-\tau_0) = \frac{(\tau-\tau_0)^q}{q!} \text{ for } q \geq 0.$$

FORMULAE FOR MULTIPOLE MOMENTS

Using the approximations of the density function we have:

- for standard straight boundary element

$$M_{LS}^{q}(\tau_0')=\frac{1}{q!}\sum_{k=1}^{3} f_k \sum_{j=0}^{2} c_{kj}\, l\left(le^{i\alpha_C}\right)^q \int_{-1}^{1} \tau'^{\,j}\,(1-\tau')^{\beta}\left(\tau'-\tau_0'\right)^q d\tau'$$

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- for standard circular-arc boundary element

$$M_{LC}^{q}(\tau_0')=\frac{1}{q!}\sum_{k=1}^{3} f_k \sum_{j=-1}^{1} \tilde{c}_{kj}\,(-iR)\left(-iR\,e^{i\alpha_C}\right)^q \int_{e^{-i\theta_0}}^{e^{i\theta_0}} \tau'^{\,j-1}\,\mathrm{Re}\left(\left(e^{i\theta_0}-\tau'\right)^{\beta}(\tau'-\tau_0')^q\right) d\tau'$$

ANALYTICAL RECURRENCE FORMULA FOR MULTIPOLE MOMENTS ON STANDARD STRAIGHT BOUNDARY ELEMENT

$$M_q^0 = \tilde{I}_q, \quad M_q^1 = \tilde{I}_{q+1} + \tau_0' \tilde{I}_q, \quad M_q^2 = \tilde{I}_{q+2} + 2\tau_0' \tilde{I}_{q+1} + \tau_0'^2 \tilde{I}_q$$

Multipoles $M_q^j = \int_{-1}^{1} w(\tau')\tau'^j \left(\tau' - \tau_0'\right)^q d\tau', \quad j = 0, 1, 2, \quad w(\tau') = (1-\tau')^\beta$ are

recurrently evaluated by using dependency: $\tau'^j = \left[\left(\tau' - \tau_0'\right) + \tau_0'\right]^j$ and analytical,

recurrent expressions for integrals $\tilde{I}_k = \int_{-1}^{1} w(\tau')\left(\tau' - \tau_0'\right)^k d\tau', \quad k = q, q+1, q+2$.

ANALYTICAL RECURRENCE FORMULA FOR MULTIPOLE MOMENTS ON STANDARD CIRCULAR-ARC BOUNDARY ELEMENT

$$M_q^0 = \tilde{I}_q, \quad M_q^{-1} = \tilde{I}_{q-1} - \tau_0' M_{q-1}^{-1}, \quad M_q^{-2} = M_{q-1}^{-1} - \tau_0' M_{q-1}^{-2}$$

For q=0, we firstly find integrals: $M_0^{-2} = \int_{e^{-i\theta_0}}^{e^{i\theta_0}} w(\tau')\frac{1}{\tau'^2}\, d\tau', \quad M_0^{-1} = \int_{e^{-i\theta_0}}^{e^{i\theta_0}} w(\tau')\frac{1}{\tau'}\, d\tau'.$

Multipoles $M_q^j = \int_{e^{-i\theta_0}}^{e^{i\theta_0}} w(\tau')\tau'^j\,(\tau'-\tau_0')^q\, d\tau',\ j = 0, -1, -2,\ w(\tau') = \mathrm{Re}\left(\left(e^{i\theta_0} - \tau'\right)^{\beta}\right)$

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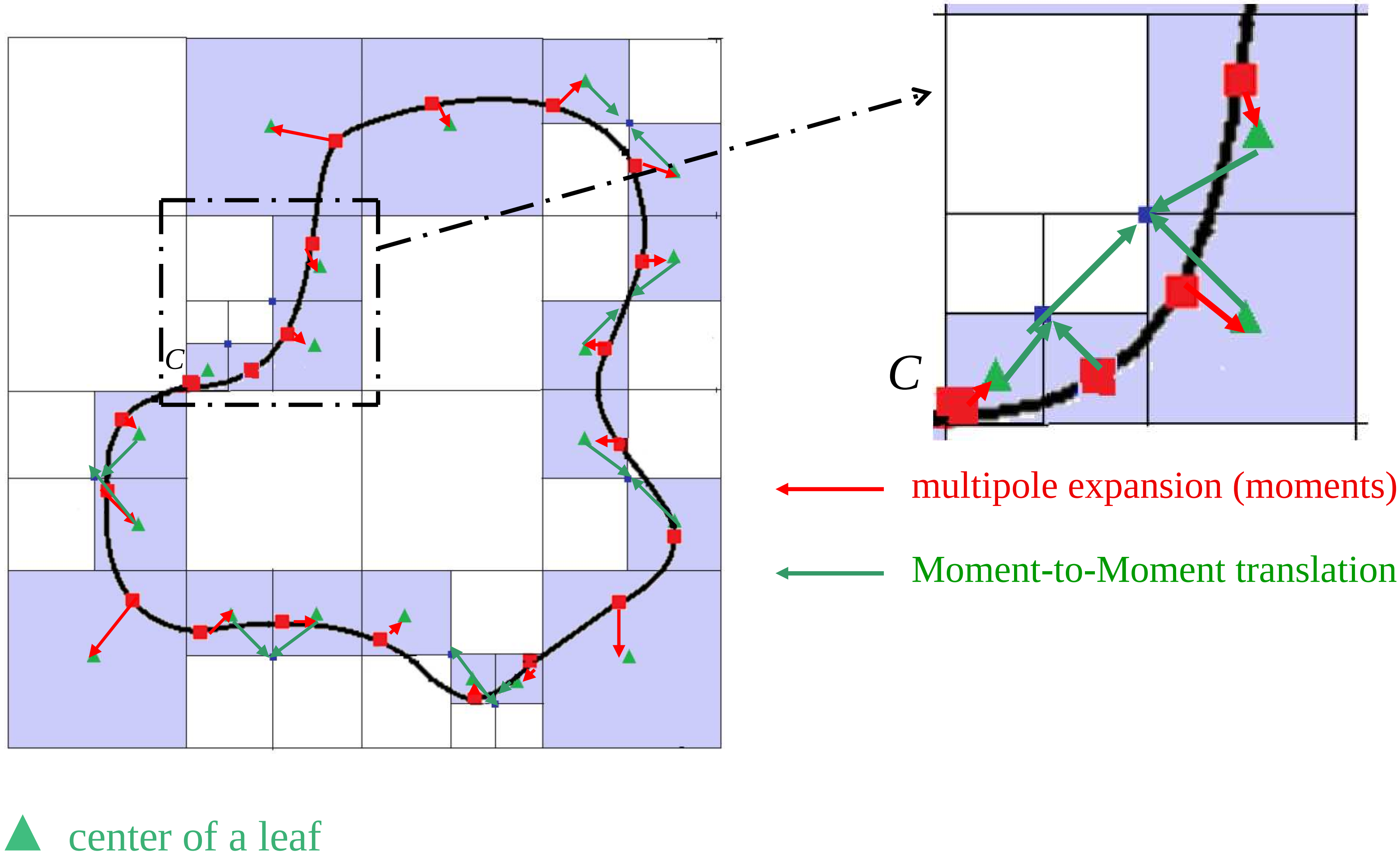
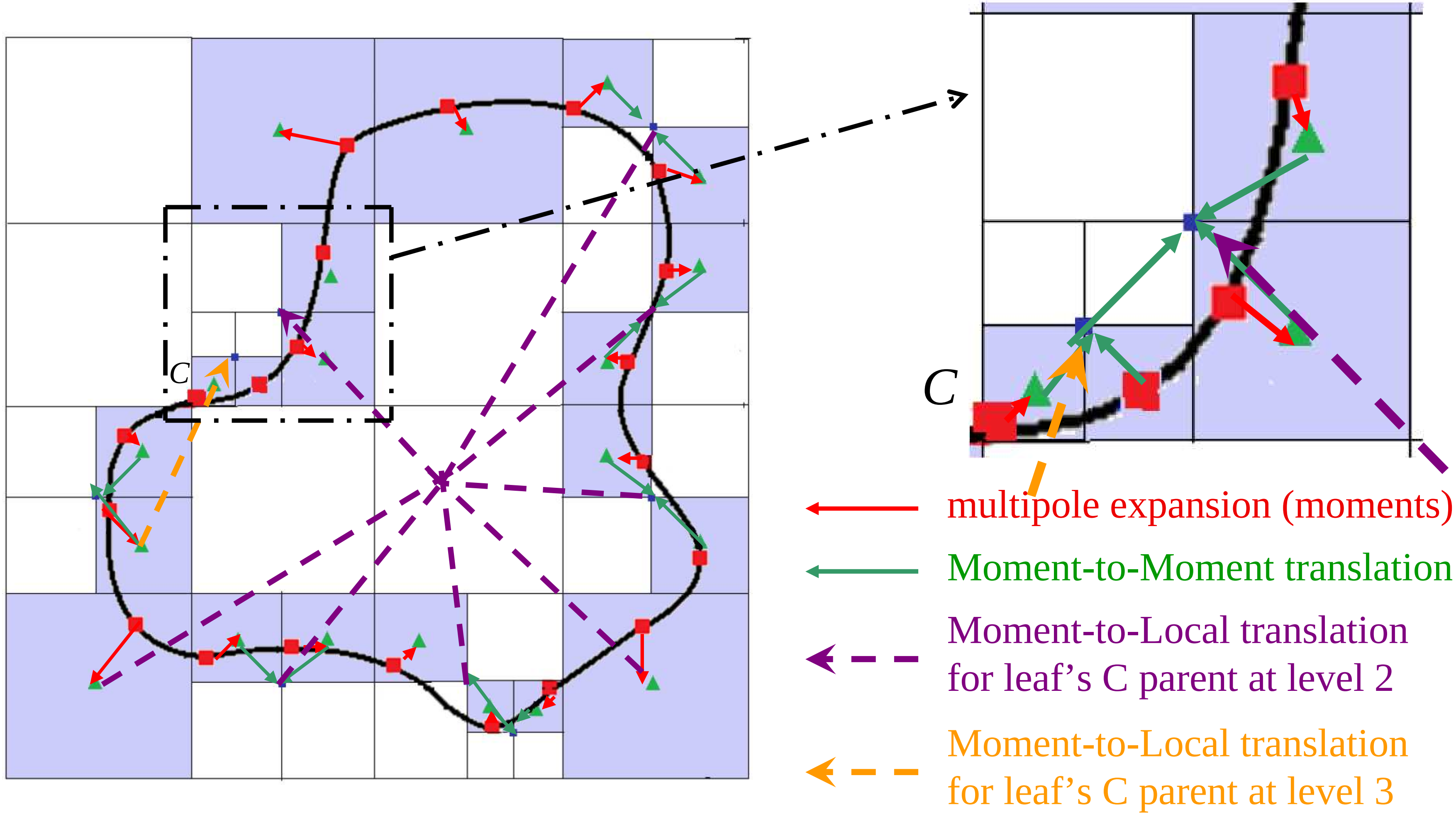
recurrent expressions for integrals: $\tilde{I}_k = \int_{e^{-i\theta_0}}^{e^{i\theta_0}} w(\tau')(\tau'-\tau_0')^k\, d\tau', \quad k = q-1, q$.

MULTIPOLE EXPANSION AND MOMENT-TO-MOMENT TRANSLATION

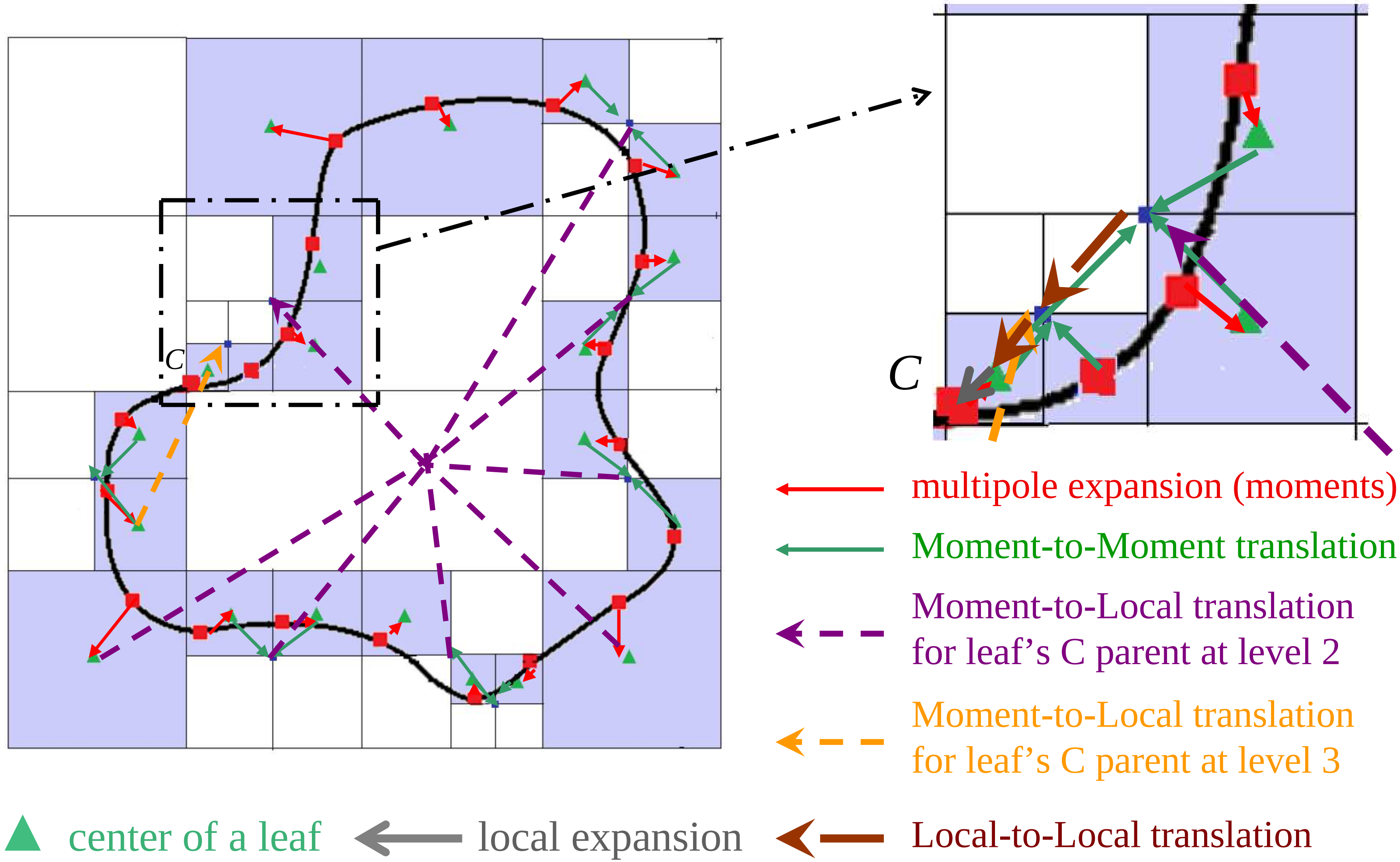


MULTIPOLE AND LOCAL TRANSLATIONS



▲ center of a leaf

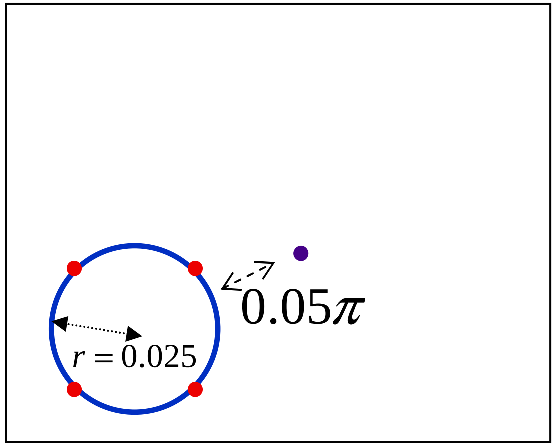
MULTIPOLE AND LOCAL TRANSLATIONS

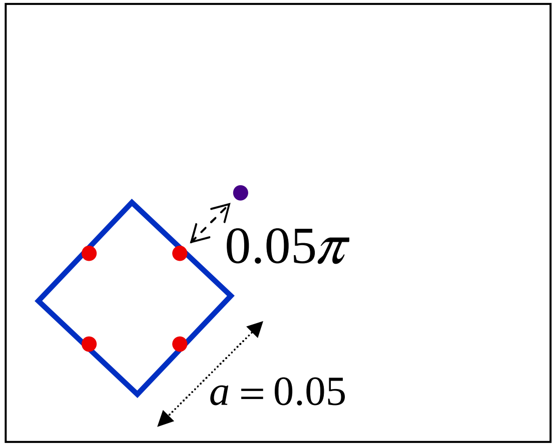


NUMERICAL EXPERIMENTS PERFORMED BY USING CV FMM

Consider a closed contour approximated by four straight or circular-arc elements.

For a field point outside the contour, the singular-type kernel integral with constant density is zero.





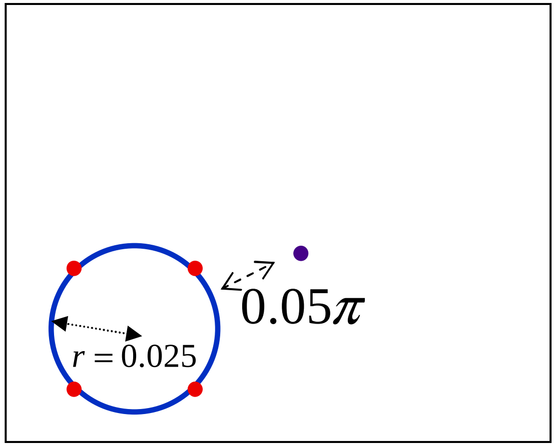
NUMERICAL EXPERIMENTS PERFORMED BY USING CV FMM

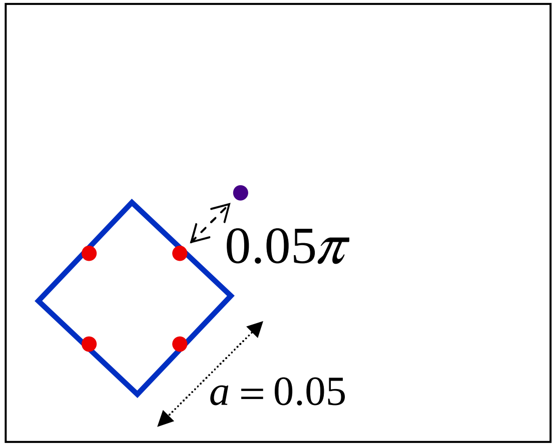
Consider a closed contour approximated by four straight or circular-arc elements.

For a field point outside the contour, the singular-type kernel integral with constant density is zero.

Application of CV FMM for the field point located outside the contour at the distance 0.05π, gives the result $10E\text{-}17$ for both types of elements.

Approximation by a larger number of elements do not affect this accuracy notably.



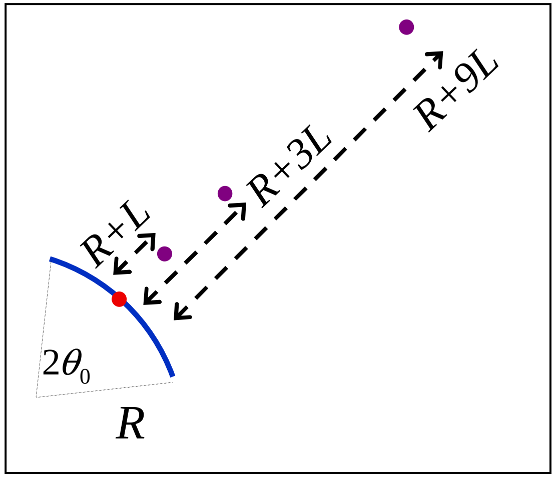


NUMERICAL EXPERIMENTS PERFORMED BY USING CV FMM

Consider a circular-arc crack with the angle $2\theta_0 = \frac{\pi}{3}$, radius $R = 0.025$ and crack length $L = 2R\theta_0$.

The distance from the center of the crack to the field point is $R + nL$, $n=1, 3, 9$. respectively.

The relative distance is: $r = \dfrac{R}{R + nL}$.



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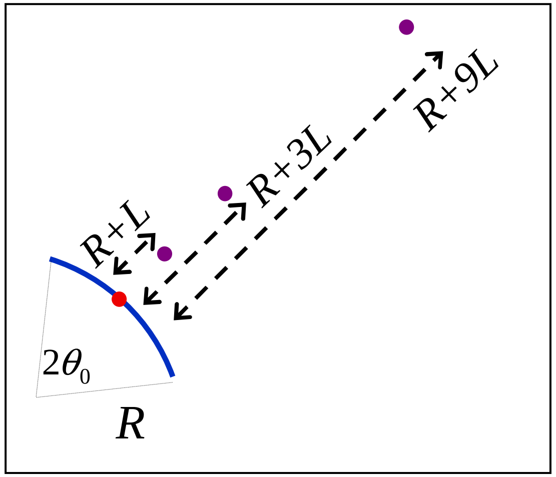

Relative error between results of direct integration and CV FMM.

maximal order of multipole moment	M_{LC}				M_{SC}			
	5	8	12	16	5	8	12	16
$r = 1/3$								
relative error	$3E-05$	$2E-05$	$7E-07$	$4E-08$	$1E-02$	$6E-04$	$4E-05$	$2E-06$
$r = 1/9$								
relative error	$1E-06$	$1E-08$	$1E-08$	$1E-08$	$1E-04$	$3E-07$	$1E-10$	$1E-13$
$r = 1/19$								
relative error	$1E-07$	$1E-08$	$1E-08$	$1E-08$	$2E-06$	$1E-09$	$1E-12$	$3E-13$

THANK YOU FOR ATTENTION