On Numerical Simulation of Hydraulic Fracturing

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Abstract

The paper concerns with numerical modeling of hydraulic fracture. It is shown that proper using of the mass balance leads to the local speed equation, which may serve for employing well-developed methods of the theory of propagating interfaces. It appears that existence of the speed equation, which is additional to a prescribed boundary condition at the liquid front, can make a problem ill-posed and complicate numerical simulation. It is also shown how the very source of the difficulty gives the key to overcome it. To this end, a method of ε-regularization is suggested. It consists in using the speed equation together with a prescribed boundary condition to formulate a new boundary condition at a small distance behind the front rather than on the front itself. Specially designed numerical experiments reveal the peculiarities of the problem. They evidently show (i) strong deterioration of the solution near the liquid front if the regularization is not employed, (ii) high accuracy and stability of the solution obtained by using the suggested regularization. We conclude that the ε-regularization presents an efficient means for numerical simulation of hydraulic fracturing.

1 Introduction

Hydraulic fracturing is widely used with the purpose to increase surface to or from which a fluid flows in rock. It is applied to enhance oil, gas and thermal heat production, to control rock pressure, to provide efficient CO_2 sequestration and to isolate contaminants. A similar process of fracturing occurs in natural conditions when pressurized magma fractures earth crust leading to formation of veins. Numerous papers have been published on mathematical modeling of hydraulic fracture starting from the works [1], [2], [3], [4],[5]. To date, authors have employed *global* mass balance to trace the liquid front propagation (e.g. [6], [7], [8], [9], [10]). In this paper, we derive and use the *local* consequence of the balance. It gives us general equation specifying the speed function for problems of hydraulic fracturing. The speed

function, being the basis of the well-developed theory of propagating interfaces, it may serve for employing level set methods and fast marching methods for hydraulic fracture simulation. Its employing also shows that when we fix the liquid contour in iterations of numerical simulation, a problem may become over-determined and consequently ill-posed. A way to overcome the difficulty and to obtain accurate and stable results follows from the very source of the difficulty. The speed equation together with a prescribed boundary condition gives us an asymptotic equation, which may be efficiently used for the problem regularization by imposing a new boundary condition at a small distance behind the front rather than on the front itself. Results of numerical experiments demonstrate the computational features of the problem and the efficiency of the regularization method.

2 Speed function and speed equation for hydraulic fracturing

For flow of incompressible liquid in a narrow channel, the global mass balance reads

$$\frac{dV_e}{dt} = \int_{S(t)} \frac{\partial w}{\partial t} dS + \int_{L(t)} q_*(x_*) dL,$$

where V_e is the volume of liquid entering/leaving the channel from/to external sources/sinks; $S(t)$ is the middle surface of the channel; w is its width (opening); $L(t)$ is the contour of the liquid at the time t; x_* is a point on the liquid front; $q_{n*}(x_*) = w_* v_{n*}$ is the total flux through the front cross section with the external normal n to the liquid front L; $w_* = w(x_*)$; $v_{n*} = dx_*/dt$ is the normal component of the front velocity. From the expression for q_{n*} we have

$$v_{n*} = \frac{q_{n*}}{w_*}. \tag{1}$$

In cases when a lag between the liquid front and the crack tip is neglected (e.g. [5], [6], [7], [8], [9], [10]), both q_{n*} and w_* are zero at the liquid front, which in these cases coincides with the tip of a crack where the flow occurs. Then the r. h. s. of (1) becomes uncertainty $\frac{0}{0}$, and equation (1) takes the form:

$$v_{n*} = \lim_{\mathbf{x} \to \mathbf{x}_*} \frac{q_{n*}}{w_*}. \tag{2}$$

The limit on the r. h. s. of (2) should be finite to exclude instant fracture propagation with infinite velocity.

The flow of viscous incompressible liquid in a narrow channel is described by partial differential equations (PDE) by Poiseuille and Reynolds. The Poiseuille equation connects the flux $\mathbf{q}$ through a cross-section with the pressure gradient as

$$\mathbf{q} = -D(w, p) \operatorname{grad} p, \tag{3}$$

where $D(w, p)$ is a prescribed function or operator, p is the pressure. Substitution of (3) into the continuity equation gives the Reynolds equation:

$$\frac{\partial w}{\partial t} - \operatorname{div}[D(w, p) \operatorname{grad} p] - q_e = 0, \tag{4}$$

where q_e is the intensity of external sources per unit area of the middle surface of the channel. Note that the PDE (3) and (4) contain the flux $\mathbf{q}$ through a cross-section rather than the particle velocity $\mathbf{v}$. The latter enters only the speed equation (1) or its limit form (2). This implies that the r.h.s. of (1) defines the so-called speed function [11] for problems of hydraulic fracturing. The speed function may serve to employ the well-established theory of propagating interfaces and methods developed in this theory, such as level set methods and fast marching methods, to simulate hydraulic fracturing.

Using (3) in (1) gives the explicit form of the speed function of hydraulic fracture:

$$v_{n*} = -\frac{1}{w_*} D(w, p) \frac{\partial p}{\partial n} \lfloor_{x=x_*}. \tag{5}$$

The speed equation (1) or its explicit form (5) is additional to a boundary condition (BC) imposed at the liquid front by considering a physical state ahead of the front. In cases when a lag between the liquid front and the crack tip is taken into account, the pressure $p_0(x_*)$ is commonly prescribed (e. g. [12]):

$$p(x_*) = p_0(x_*). \tag{6}$$

In cases, when the lag is neglected, the opening at the crack tip is set zero:

$$w(x_*) = 0. \tag{7}$$

The opening being unknown, a solid mechanics equation is used to connect it with the pressure. Besides, a fracture mechanics condition is imposed to let the fracture surface change in time. Because of using these equations, the condition (7), like (6), actually involves the pressure. For certainty, in further discussion, we consider the BC (6).

In view of (6) and (5), we have two rather than one BC for the PDE (4) at the points of the liquid front. The first of them is the prescribed BC (6), while the second is the speed equation (5). Note now that the operator $div[D(w, p)gradp]$ is elliptic in spatial coordinates. Thus, when solving numerically a problem involving (4) under a fixed contour $L(t)$ at a time step, we have a Cauchy (initial value) problem for an elliptic operator. Consequently, a problem may appear ill-posed [13], [14]. This strongly complicates obtaining accurate and stable numerical results. To overcome the difficulty, it is reasonable to have an appropriate method of regularization.

3 Regularization method

We suggest the regularization method based on employing the very cause of the difficulty. Specifically, the equations (5) and (6) are used together to formulate a new BC at a small distance behind the front rather than on the front itself.

Consider a point x_* at the front and introduce the local Cartesian coordinates $x_1' O' x_2'$ with the origin O' at the point x_*, the axis x_1' opposite to the direction of the normal to the front and the axis x_2' in the tangent direction. The flux $\mathbf{q}$ is normal to the front, hence, its tangential component is zero, while according to (3) its normal

component is $q_{n*} = D(w,p)\frac{\partial p}{\partial x_1'}$. Then (5) yields the approximate equation for points at a vicinity of the front:

$$\frac{1}{w}D(w,p)\frac{\partial p}{\partial x_1'} \approx v_*, \tag{8}$$

where $v_* = dx_*/dt$ is the absolute value of the front velocity. Integration of (8) and accounting for (6) gives:

$$\int_{p_0}^{p} \frac{1}{w}D(w,p)dp \approx v_* x_1'. \tag{9}$$

By using (9) we may impose the BC at a small distance $x_1' = \varepsilon'$ behind the front:

$$\int_{p_0}^{p_\varepsilon} \frac{1}{w}D(w,p)dp = v_*\varepsilon', \tag{10}$$

where p_ε is the pressure at the point ε'. Thus instead of using the BC (6) at the front, we suggest to employ the regularized condition (10) behind the front. It is obtained by integrating the speed equation in the local coordinates and by using a prescribed BC when evaluating the lower limit of the integral. Besides, in numerical calculations, taking into account that the distance ε' is small, the equation (8) is written as the equality at the point ε':

$$\frac{1}{w}D(w,p)\frac{\partial p}{\partial x_1'}\lfloor_{x_1'=\varepsilon'} = v_*. \tag{11}$$

Equation (11) may serve for iterations in v_* when employing finite differences to numerically simulate fracture propagation.

4 Numerical tests without and with regularization

We tested the regularization method by revisiting the Nordgren problem [5] in the case when there is no leak-off ($q_e = 0$). In the model used by Nordgren, 1-D flow of incompressible Newtonian liquid occurs along the interval $[0, x_*(t)]$ of the x-axis; the pressure p is proportional to the opening w and the lag is zero, so that both the opening and the flux are zero at the liquid front (the pressure is also zero at the front because it is proportional to the opening). There is a prescribed flux q_0 at the inlet $x = 0$. After normalizing variables, the mathematical problem is formulated as follows [5]: we need to solve the Reynolds PDE, which takes the form

$$\frac{\partial^2 w^4}{\partial x^2} - \frac{\partial w}{\partial t} = 0 \tag{12}$$

under the boundary conditions at the inlet

$$\frac{\partial w^4}{\partial x}\lfloor_{x=0} = q_0 \tag{13}$$

and at the liquid front

$$w(x_*, t) = 0. \tag{14}$$

The opening $w(x, t)$ is assumed positive behind the front and zero ahead of it:

$$w(x, t) > 0 \ \text{ for } 0 \le x < x_*; \ \ w(x, t) = 0 \ \text{ for } x > x_*. \tag{15}$$

Note that the last equations actually serve to find the front velocity as a part of the solution.

At the initial moment $t = 0$, the opening is zero along the entire x-axis:

$$w(x, 0) = 0. \tag{16}$$

Nordgren solved the problem (12)-(16) by using the Crank-Nicolson scheme and linear approximation for w^4. The results obtained are given in [5] to the accuracy of some one percent. Our attempts to reach higher accuracy without regularization failed: *by no means could we obtain more than two significant digits*. Using various variables (w, w^3, w^4) and employing various formulations of the problem, including the self-similar formulation, made not good. Remarkable is that when using fine meshes, the results always strongly deteriorated near the liquid front. This evidently confirms the theoretical proposition that the problem is ill-posed and needs proper regularization to get accurate and stable numerical results. As noted above, the reason is that the speed equation is additional to a prescribed boundary condition at the liquid front.

In the case considered the speed equation reads:

$$-\frac{4}{3}\frac{\partial w^3}{\partial x}\Big|_{x=x_*} = v_*(t). \tag{17}$$

Following the line sketched in the previous section, we use (14) and (17) together to impose the boundary condition at a small distance ε' behind the front:

$$w^3\big|_{x=x_*-\varepsilon'} = \frac{3}{4} v_*(t)\varepsilon'. \tag{18}$$

Equation (11), serving for iterations in v_* at a time step, becomes:

$$\frac{\partial w^3}{\partial x}\Big|_{x=x_*-\varepsilon'} = -\frac{3}{4} v_*(t). \tag{19}$$

We solved the regularized problem (12), (13), (18) by Crank-Nicolson method with iterations in non-linear terms and in v_* (according to (19)). In contrast with the solution without regularization, the results obtained with regularization are accurate and stable. Specifically, we have at least five correct significant digits when taking the relative distance $\varepsilon = \varepsilon'/x_*$ from the front as small as 10^{-3}, 10^{-4}, the spatial step $\Delta\zeta = \Delta x/x_*$ in the range $10^{-6} \le \Delta\zeta \le 10^{-3}$ and sufficiently small time step ($\Delta t < t_0$, where t_0 is the starting time). No signs of instability were observed in specially designed numerical experiments with the number of time steps exceeding 20000. The method deteriorated only when the spatial step $\Delta\zeta$ and

the regularization parameter ε were very small *simultaneously*: $\Delta\zeta = \varepsilon = 10^{-6}$. Actually, in this case, to the employed accuracy of double precision, there is no regularization (ε is too small), while the mesh is very fine ($\Delta\zeta$ is too small); according to the discussion above, calculations without regularization deteriorate when using very fine meshes. The time expense on a conventional computer never exceeded 15 seconds. We conclude that the suggested regularization is efficient.

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