Evaluation of temporal derivative for propagating front of hydraulic fracture

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Abstract

The work concerns with the problem of hydraulic fracture propagating in time. Then in the hypersingular equation, connecting the net-pressure with the fracture opening, the integrand and limits of integration depend on the parameter (time). The rate of the pressure change, being of practical significance, we derive a rule for evaluation of the time derivative of a hypersingular integral with respect to a parameter. We present (i) the new concept of the complex variable hypersingular (CVH) integral with the density and limits of integration depending on a parameter, (ii) the theorem, which gives the rule for differentiation of the integral with respect to the parameter, (iii) application of the derived rule to the particular case when the fracture propagates under constant net-pressure.

1 Introduction

Using complex variable (CV) singular and hypersingular integral equations has proved to be an efficient means for solving various problems of fluid and solid mechanics. In particular, they are applied when studying hydraulic fractures (e.g. [1], [2], [3]). To the date, the theory of the CV singular [4] and hypersingular [5] integrals, refers to problems, in which the boundary of a surface is fixed. Meanwhile, in problems of hydraulic fractures the boundary of the fracture propagates in time. Therefore, when considering hydraulic fractures, we need to extend the theory and to obtain a rule, which allows one to perform the differentiation of the CV singular and hypersingular integrals with respect to a parameter (time) when the density and/or contour depend on the parameter.

The main result of the paper is expressed by the proved theorem, which states that under physically sound assumptions, the usual rule of differentiation of a proper integral with respect to a parameter stays true for CVH integrals of arbitrary order. The paper contains also needed prerequisites and illustration of the derived rule by the example of the hydraulic fracture, propagating at early stage after initiation, when the net-pressure is actually constant along the fracture surface.

2 Problem formulation

A mathematical formulation of the problem of hydraulic fracture includes (i) fluid, (ii) solid, and (iii) fracture mechanics equations (see, e.g. [1], [2]). In this paper we focus on the second group as that, which defines the dependence between the net-pressure p and the fracture opening w. In the simplest case of a 2D problem for a straight crack, propagating along the x-axis, the dependence is given by the classical equation [6]:

$$p(x) = -\frac{E}{4\pi(1-\nu^2)} \int_a^b \frac{\partial w(\tau)}{\partial \tau} \frac{d\tau}{\tau - x}, \qquad a \leq x \leq b, \tag{1}$$

where E is the elasticity modulus, ν is the Poisson's ratio of the rock mass, a and b are points corresponding to the edges of the fracture; the integral on the r.h.s. is assumed as the singular (principle value) integral. The equation (1) contains the spatial derivative of the opening rather than the opening itself what is inconvenient in practical calculations. Thus it is reasonable to re-write (1) in the hypersingular form:

$$p(\alpha, x) = -\frac{E}{4\pi(1-\nu^2)} \int_{a(\alpha)}^{b(\alpha)} \frac{w(\alpha, \tau) d\tau}{(\tau - x)^2}, \tag{2}$$

where we have also taken into account that for a propagating fracture, its edges a, b, the opening w and the net-pressure p are functions of the time. Thus, equation (2) defines the change of the net-pressure in time as a hypersingular integral with the density and limits depending on a parameter α. The latter, in the considered problem, is the time. The rate of the pressure change $\partial p(\alpha, x)/\partial \alpha$ is a characteristic strongly dependent on the fluid injection regime. Its evaluation is also of need for numerical modeling of hydraulic fractures. Therefore, it is reasonable to obtain a rule for evaluation of the derivative of the hypersingular integral in (2) with respect to the parameter α. To get such a rule, we employ and extend the general theory of CV hypersingular integrals, presented in [5].

3 Theorem on the derivative with respect to parameter

Let ab be an open curve (arc) in the complex plane $z = x + iy$ ($i = \sqrt{-1}$). The equation of the arc is $\tau(\gamma) = x(\gamma) + iy(\gamma)$, where γ is a real parameter such that its value γ_a corresponds to start point a, while the value γ_b corresponds to end point b: $a = x(\gamma_a) + iy(\gamma_a)$, $b = x(\gamma_b) + iy(\gamma_b)$. The arc is smooth in the sense explained in [4]. In further discussion, the positions of the edges a and b may change depending on a real parameter α. Thus $\gamma_a = \gamma_a(\alpha)$, $\gamma_b = \gamma_b(\alpha)$, $a = a(\alpha)$, $b = b(\alpha)$. We assume that the functions $\gamma_a(\alpha)$, $\gamma_b = \gamma_b(\alpha)$ have Holder continuous derivatives.

Consider a hypersingular integral of order k

$$I_k(\alpha, t) = \int\limits_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha, \tau)}{(\tau - t)^k} d\tau \tag{3}$$

with the density $g(\alpha, \tau)$ depending on the parameter α. We assume that the density has Holder continuous $k-th$ derivative $\frac{\partial^k g(\alpha,\tau)}{\partial \tau^k}$ with respect to τ for each α and it also has Holder continuous derivative $\frac{\partial g(\alpha,\tau)}{\partial \alpha}$ with respect to α for each $\tau \in ab$. For any fixed α and $t \in ab$, the integral (3) is defined in accordance with the general theory [5]. Consequently, for a fixed α, the following formulae, used below, are true.

(i) Extended Newton-Leibnitz formula:

$$\int_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha, \tau)}{(\tau - t)^k} d\tau = J_g(\alpha, b) - J_g(\alpha, a) + \frac{i\pi}{k!} g_t^{(k-1)}(\alpha, t), \tag{4}$$

where $J_g(\alpha, \tau)$ is an antiderivative of the integrand $\frac{g(\alpha,\tau)}{(\tau-t)^k}$, that is

$$\frac{\partial J_g(\alpha, c)}{\partial c} = \frac{g(\alpha, c)}{(c - t)^k}. \tag{5}$$

(ii) The third regularization formula for $k \geq 2$:

$$\frac{d}{dt} \int\limits_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha, \tau)}{(\tau - t)^{k-1}} d\tau = (k-1) \int\limits_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha, \tau)}{(\tau - t)^k} d\tau. \tag{6}$$

By differentiating (5) with respect to α and changing the order of derivatives with respect to α and c, what is justified under accepted assumptions, we obtain:

$$\frac{\partial}{\partial c} \left(\frac{\partial J_g(\alpha, c)}{\partial \alpha} \right) = \frac{\frac{\partial g(\alpha,c)}{\partial \alpha}}{(c - t)^k}. \tag{7}$$

Equation (7) means that $\frac{\partial J_g(\alpha,c)}{\partial \alpha}$ is an antiderivative of the function $\frac{\partial g(\alpha,c)/\partial \alpha}{(c-t)^k}$. Then using this function and its antiderivative in the extended Newton-Leibnitz formula (4), we obtain:

$$\int_{a(\alpha)}^{b(\alpha)} \frac{\frac{\partial g(\alpha,\tau)}{\partial \alpha}}{(\tau - t)^k} d\tau = \frac{\partial J_g}{\partial \alpha}(\alpha, b) - \frac{\partial J_g}{\partial \alpha}(\alpha, a) + \frac{i\pi}{k!} \frac{\partial g_t^{(k-1)}}{\partial \alpha}(\alpha, t). \tag{8}$$

On the other hand, the results of differentiation of the both parts of (4) with respect to α may be written as

$$\frac{\partial}{\partial \alpha} \int_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha, \tau)}{(\tau - t)^k} d\tau = \frac{\partial J_g}{\partial \alpha}(\alpha, b) - \frac{\partial J_g}{\partial \alpha}(\alpha, a) + \frac{i\pi}{k!} \frac{\partial g_t^{(k-1)}}{\partial \alpha}(\alpha, t) +$$

$$+ \frac{\partial J_g}{\partial b} \frac{db}{d\alpha} - \frac{\partial J_g}{\partial a} \frac{da}{d\alpha}. \tag{9}$$

Noting that the sum of the first three terms on the r.h.s. of (9) is given by the integral in (8), equation (9) becomes:

$$\frac{\partial}{\partial\alpha}\int_{a(\alpha)}^{b(\alpha)}\frac{g(\alpha,\tau)}{(\tau-t)^k}d\tau=\int_{a(\alpha)}^{b(\alpha)}\frac{\partial g(\alpha,\tau)}{\partial\alpha}\frac{d\tau}{(\tau-t)^k}+\frac{g(\alpha,b)}{(b-t)^k}\frac{db}{d\alpha}-\frac{g(\alpha,a)}{(a-t)^k}\frac{da}{d\alpha}. \quad (10)$$

Equation (10) shows that the classical rule for differentiation of a proper integral with respect to a parameter holds for a CV hypersingular integral of an arbitrary order, as well. Thus, we have proved the theorem.

Theorem (Differentiation of a CVHI with respect to a parameter). *For a smooth arc* a, b *with* $a(\alpha)$ *and* $b(\alpha)$ *being Holder continuous in a parameter* α *and for a density* $g(\alpha,\tau)$ *having* $k-1-$*th Holder continuous derivative with respect to* τ *and Holder continuous derivative with respect to* α*, the derivative of a hypersingular integral* $I_k(\alpha,t)$ *with respect to the parameter* α *has the form (10) similar to the common rule for proper integrals.*

In the problem of hydraulic fracturing, $k = 2$, α has the meaning of the time, the integral on the l.h.s. of (10) is proportional to the net-pressure, the density $g(\alpha,\tau)$ is the fracture opening and the derivatives $db/d\alpha$ and $da/d\alpha$ express the speeds, with which the fracture front propagates. According to (10), the inflence of the speeds on the rate of the pressure change strongly depends on the values $g(\alpha,a)$ and $g(\alpha,b)$ of the opening at the points of the front a and b. Usually, near a point c of the front, the opening tends to zero as $(c-\tau)^\gamma$, where $\gamma > 0$. Hence, we need to extend the theorem to the case when near an edge point c ($c = a$ or $c = b$) the density is of the form $g(\alpha,\tau) = (c-\tau)^\gamma g_\gamma(\alpha,\tau)$.

4 Extension to densities with derivatives having power-type singularity at arc tips

Consider a density of the form $g(\alpha,\tau) = (c-\tau)^\gamma g_\gamma(\alpha,\tau)$. For generality, we assume that γ is a complex number with $\mathrm{Re}\gamma > 0$. Note that if $j-1 < \mathrm{Re}\gamma < j$, where j is non-negative integer, then the derivatives $\partial^j g(\alpha,\tau)/\partial\tau^j$ and $\partial^j g(c,\tau)/\partial\tau^{j-1}\partial c$ are singular at the point $\tau = c$, tending to infinity as $1/(c-\tau)^{j-\mathrm{Re}\gamma}$. As the definitions of the hypersingular integral and the theorem of the previous section employ assumptions on the derivatives, there is need in further agreements on the behaviour of the density. We shall assume that $k-2 < \mathrm{Re}\gamma < k-1$ and call $g(\alpha,\tau) = (c-\tau)^\gamma g_\gamma(\alpha,\tau)$ the density of class H_*^k. For $k = 1$, the class H_*^k coincides with the class H_*, defined and studied in [4].

For the density of class H_*^k, we may represent the CVHI (3) as the sum of three integrals

$$\int_{a(\alpha)}^{b(\alpha)}\frac{g(\alpha,\tau)}{(\tau-t)^k}d\tau=\int_{a_1(\alpha)}^{b_1(\alpha)}\frac{g(\alpha,\tau)}{(\tau-t)^k}d\tau+\int_{a(\alpha)}^{a_1(\alpha)}\frac{g(\alpha,\tau)}{(\tau-t)^k}d\tau+\int_{b_1(\alpha)}^{b(\alpha)}\frac{g(\alpha,\tau)}{(\tau-t)^k}d\tau, \quad (11)$$

where $a_1(\alpha)$ is an arbitrary point between $a(\alpha)$ and t, while $b_1(\alpha)$ is an arbitrary point between t and $b(\alpha)$. The first of them does not contain the edges as points of integration; hence the general theory and the theorem are applicable to it. Two remaining integrals are usual improper integrals because the point t does not belong to their intervals of integration; their partial derivatives with respect to α may be evaluated in a common way because, under the assumptions, the partial derivative $\partial g(c,\tau)/\partial c$ is integrable. This implies the extension of the theorem.

Extended theorem: *for a density of class* H^k_*, *the theorem holds for points within an open arc* ab.

We shall not dwell on the limit values of the derivative $\partial I_k/\partial\alpha$, when $t \to c$ ($c = a$ or $c = b$). They require involved calculations and will be discussed in a paper in preparation.

For $k \geq 2$, we have $\mathrm{Re}\gamma > 0$. Consequently, the density is zero at the edge points: $g(\alpha, c) = 0$. Hence, in this case, the differentiation formula (10) means that it is possible to differentiate under the integral sign:

$$\frac{\partial}{\partial\alpha}\int_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha,\tau)}{(\tau-t)^k} d\tau = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial g(\alpha,\tau)}{\partial\alpha}\frac{d\tau}{(\tau-t)^k}. \tag{12}$$

This result is of special significance for hydraulic fractures, because the opening is zero at the fracture front. In view of the regularization formula (6), used in the form similar to (11), equation (12) may be written as

$$\frac{\partial}{\partial\alpha}\int_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha,\tau)}{(\tau-t)^k} d\tau = \frac{1}{k-1}\frac{\partial}{\partial t}\int_{a(\alpha)}^{b(\alpha)} \frac{\partial g(\alpha,\tau)}{\partial\alpha}\frac{d\tau}{(\tau-t)^{k-1}}. \tag{13}$$

In the next section, we shall check this formula by separate evaluation its left and right hand side for a particular case of the hydraulic fracture propagation.

5 Example

At an early stage of the hydraulic fracturing, the fracture propagates in the toughness dominated regime, when the influence of viscosity is negligible and the net-pressure is actually constant along the fracture: $p = p(\alpha), \partial p/\partial x = 0$ (recall that in the considered problem, parameter α is the time, x is the spatial coordinate). Then for plain-strain conditions, the opening w in (1) is given by the well-known formula (e.g. [6]):

$$w(\alpha,\tau) = \frac{4(1-\nu^2)}{E}p(\alpha)\sqrt{[\tau-a(\alpha)][b(\alpha)-\tau]},$$

where we have taken into account that p, a and b may change in time depending on the injection regime and local changes of fracture toughness. The equation (2) becomes:

$$p(\alpha) = -\frac{1}{\pi}\int_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha,\tau)d\tau}{(\tau-x)^2}, \tag{14}$$

with $g(\alpha,\tau) = p(\alpha)\sqrt{[\tau - a(\alpha)][b(\alpha) - \tau]}$. For the derivative $\partial p/\partial\alpha = dp/d\alpha$, it yields

$$\frac{dp}{d\alpha} = -\frac{1}{\pi}\frac{\partial}{\partial\alpha}\int\limits_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha,\tau)d\tau}{(\tau - x)^2}. \tag{15}$$

We want to evaluate the r.h.s. of (15) by employing the derived formula (13) and to compare the result with the l.h.s. of (15). To this end, the next two identities, easily following from the general theory [4], [6] for $x \in ab$, are used:

$$\int\limits_a^b \frac{\sqrt{(\tau - a)(b - \tau)}d\tau}{(\tau - x)^2} = -\pi, \quad \int\limits_a^b \frac{d\tau}{\sqrt{(\tau - a)(b - \tau)}(\tau - x)} = 0. \tag{16}$$

The first of them, actually gives (14) for the considered $g(\alpha,\tau)$.

With $g(\alpha,\tau) = p(\alpha)\sqrt{[\tau - a(\alpha)][b(\alpha) - \tau]}$, $t = x$ and $k = 2$, the differentiation rule (13) yields:

$$\frac{\partial}{\partial\alpha}\int_{a(\alpha)}^{b(\alpha)} \frac{p(\alpha)\sqrt{[\tau - a(\alpha)][b(\alpha) - \tau]}}{(\tau - x)^2}d\tau =$$

$$= \frac{dp(\alpha)}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} \frac{\sqrt{[\tau - a(\alpha)][b(\alpha) - \tau]}d\tau}{(\tau - x)^2} + p(\alpha)\frac{1}{2}\frac{\partial}{\partial x}\int\limits_a^b \frac{(\tau - a)db/d\alpha - (b - \tau)da/d\alpha}{\sqrt{(\tau - a)(b - \tau)}(\tau - x)}d\tau.$$

By using (16) and taking into account that a and b depend only on the time α, we have

$$\frac{\partial}{\partial\alpha}\int_{a(\alpha)}^{b(\alpha)} \frac{p(\alpha)\sqrt{[\tau - a(\alpha)][b(\alpha) - \tau]}}{(\tau - x)^2}d\tau =$$

$$= -\pi\frac{dp(\alpha)}{d\alpha} + p(\alpha)\frac{1}{2}\left(\frac{db}{d\alpha} + \frac{da}{d\alpha}\right)\frac{\partial}{\partial x}\int\limits_a^b \frac{\tau}{\sqrt{(\tau - a)(b - \tau)}(\tau - x)}d\tau.$$

Writing $\tau = (\tau - x) + x$, the last integral is represented by the sum

$$\int\limits_a^b \frac{\tau}{\sqrt{(\tau - a)(b - \tau)}(\tau - x)}d\tau = \int\limits_a^b \frac{d\tau}{\sqrt{(\tau - a)(b - \tau)}} + x\int\limits_a^b \frac{d\tau}{\sqrt{(\tau - a)(b - \tau)}(\tau - x)}.$$

The first term of the sum does not depend on x, while the second term is zero by the second of (16). Finally, we obtain:

$$\frac{\partial}{\partial\alpha}\int_{a(\alpha)}^{b(\alpha)} \frac{p(\alpha)\sqrt{[\tau - a(\alpha)][b(\alpha) - \tau]}}{(\tau - x)^2}d\tau = -\pi\frac{dp(\alpha)}{d\alpha},$$

what actually coincides with (15).

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