UNIVERSITÀ DEGLI STUDI DI TRENTO

Facoltà di Ingegneria

Italian/British Workshop on Fluid & Solids Interaction and Fracture & Failure of Solids and Structures

16 August 2011

Second order homogenization with Second Gradient Model (Mindlin)

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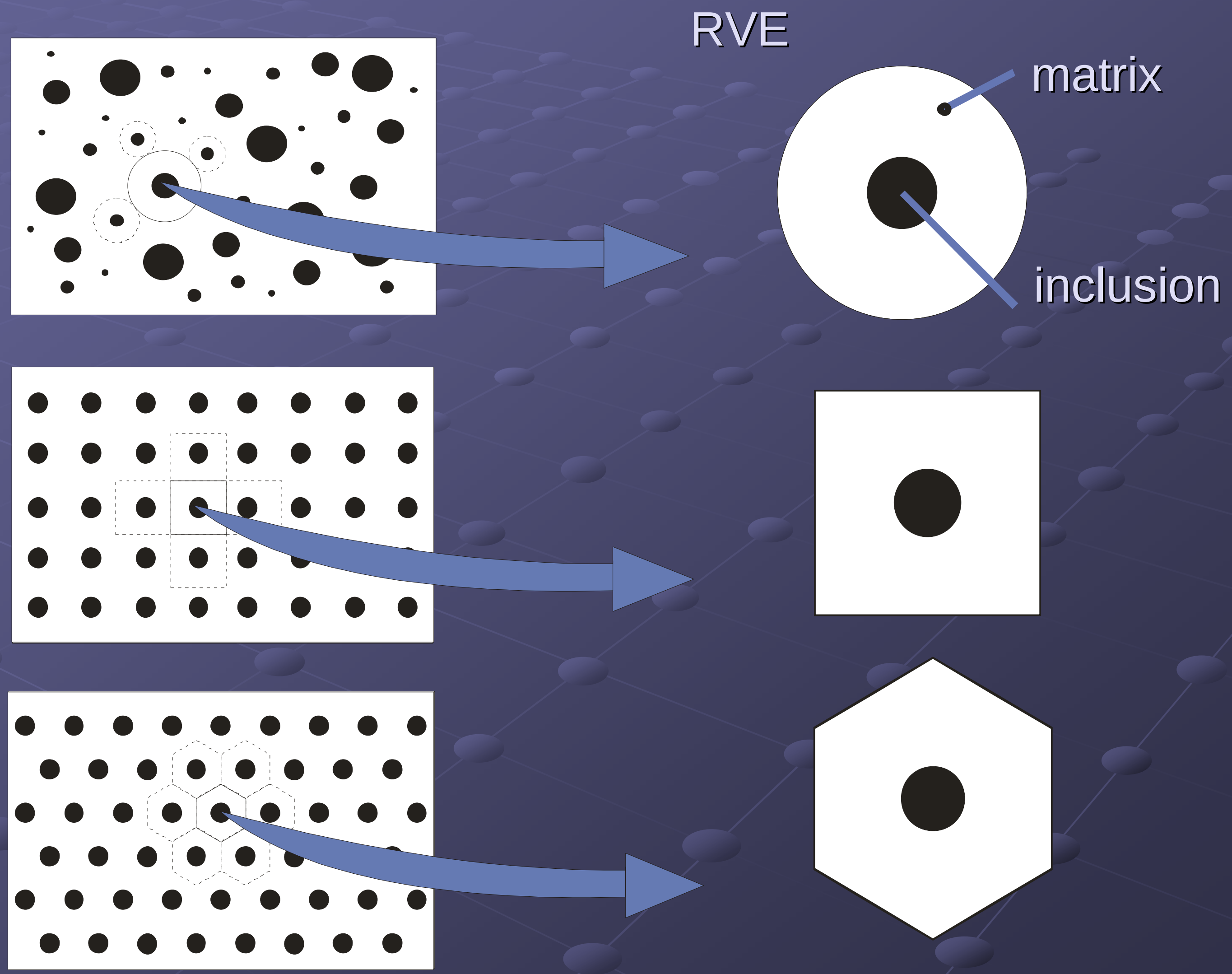
Simplest heterogeneous model: inclusion

The simplest model of heterogeneous material is the matrix-(small)inclusion model

Random inclusions:

Square matrix inclusions:

Exagonal matrix inclusions:

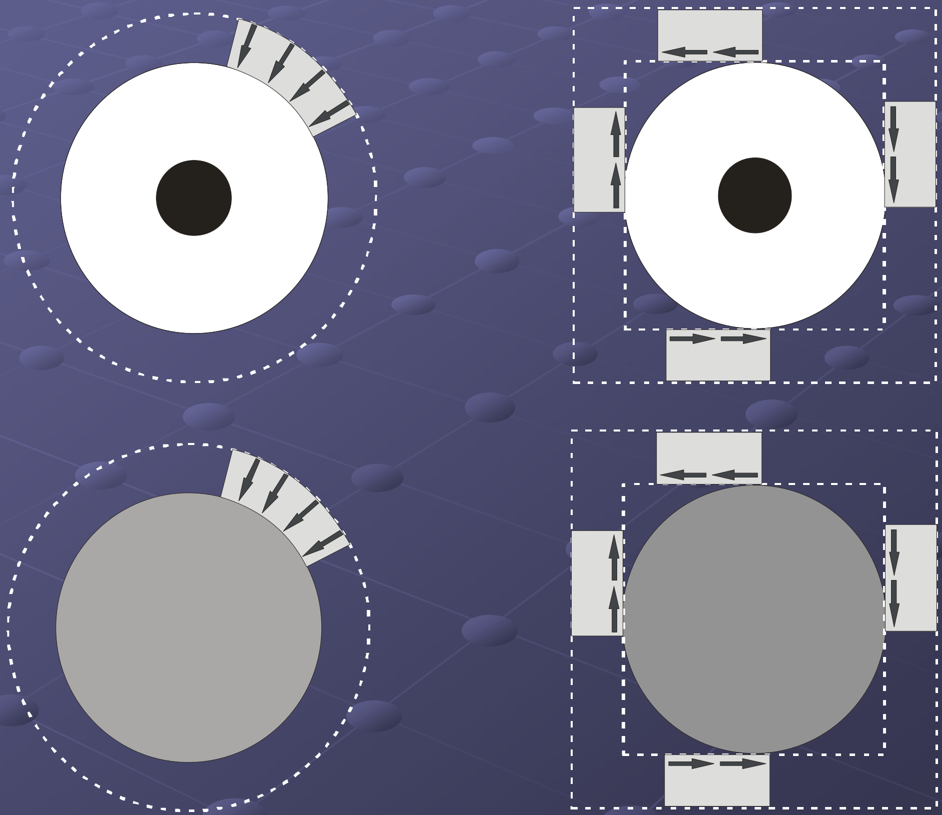


First order homogenization

Energy matching by imposing the same first order displacement field

Isotropic pressure | Shear

Heterogeneus material: μ_i, k_i ; μ_m, k_m



Homogeneus material: $\overline{\mu}, \overline{k}$

Energy matching:

$$\xi_P\left(\mu_m, k_m, \mu_i, k_i\right) - \xi_P^o\left(\overline{\mu}, \overline{k}\right) = 0$$

$$\xi_T\left(\mu_m, k_m, \mu_i, k_i\right) - \xi_T^o\left(\overline{\mu}, \overline{k}\right) = 0$$

First order homogenization

So we obtain the elastic constants of the homogenized material $\overline{\mu}, \overline{k}$

$$\bar{\mu} = \mu_m + f(1+\kappa_m)\mu_m \frac{\mu_i - \mu_m}{\kappa_m \mu_i + \mu_m}$$

$$\bar{\kappa} = \kappa_m + f(1+\kappa_m)\left[(\kappa_m - 1)\frac{\mu_i - \mu_m}{\kappa_m \mu_i + \mu_m} - \frac{(\kappa_m - 1)\mu_i - (\kappa_i - 1)\mu_m}{2\mu_i + (\kappa_i - 1)\mu_m} \right]$$

f Volume ratio of the inclusion

Notice that we don't see any shape effect into that result as long as the inclusion is diluited (f is infinitesimal)

Second order homogenization

The generic second order displacement field can be written as $u_i = \frac{\bar{u}_{i,jk}}{2}\, x_j\, x_k$

The strain becomes $\varepsilon_{is} = \frac{u_{i,s} + u_{s,i}}{2} = \frac{\bar{u}_{i,sj} + \bar{u}_{s,ij}}{2}\, x_j$

The elastic potential

$$\begin{aligned}\varphi(\varepsilon) &= \frac{1}{2}\left[\lambda\,(\varepsilon_{kk}\varepsilon_{jj}) + 2\mu\,(\varepsilon_{is}\,\varepsilon_{is})\right] \\ &= \frac{1}{2}\left[\lambda\,(\bar{u}_{i,ij}\, x_j\ \bar{u}_{k,ks}\, x_s) + 2\mu\left(\frac{\bar{u}_{i,sj} + \bar{u}_{s,ij}}{2}\, x_j\ \frac{\bar{u}_{i,st} + \bar{u}_{s,it}}{2}\, x_t\right)\right]\end{aligned}$$

The elastic energy

V is $[-\frac{h}{2}, \frac{h}{2}]^3$

$$\begin{aligned}\xi^{Cauchy} &= \int_V \varphi(\varepsilon)dV \\ &= \frac{1}{2}\left(\lambda\,\bar{u}_{i,ij}\,\bar{u}_{k,ks}\int_V x_j\, x_s\, dV + 2\mu\frac{\bar{u}_{i,sj} + \bar{u}_{s,ij}}{2}\ \frac{\bar{u}_{i,st} + \bar{u}_{s,it}}{2}\int_V x_j\, x_t\, dV\right) \\ &= \frac{1}{2}\left(\lambda\,\bar{u}_{i,ij}\,\bar{u}_{k,ks}\,\frac{h^5}{12}\,\delta_{js} + 2\mu\frac{\bar{u}_{i,sj} + \bar{u}_{s,ij}}{2}\ \frac{\bar{u}_{i,st} + \bar{u}_{s,it}}{2}\ \frac{h^5}{12}\,\delta_{jt}\right) \\ &= \frac{1}{2}\,\frac{h^5}{12}\left(\lambda\,\bar{u}_{i,ij}\,\bar{u}_{k,kj} + 2\mu\,\frac{\bar{u}_{i,js}^2 + \bar{u}_{s,ij}^2 + 2\,\bar{u}_{i,sj}\,\bar{u}_{s,ij}}{4}\right) \\ &= \frac{h^5}{12}\left[\frac{\lambda}{2}\,(\bar{u}_{i,ij}\,\bar{u}_{k,kj}) + \frac{\mu}{2}(\bar{u}_{i,js}^2 + \bar{u}_{i,js}\,\bar{u}_{s,ji})\right]\end{aligned}$$

Second order homogenization

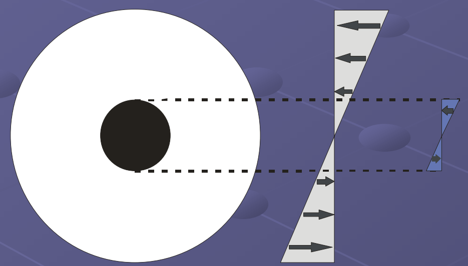
We impose a displacement field to either the heterogeneous and the homogeneous material and match the second order average strain

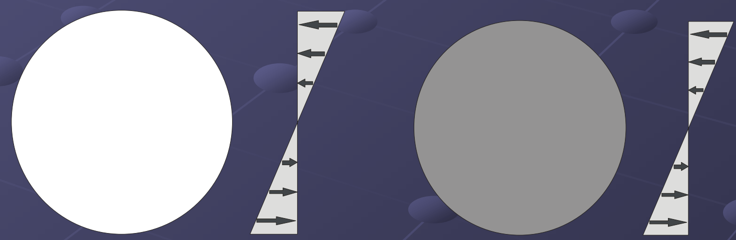
$$< \varepsilon_{is} >_k = \int_V x_k \, \varepsilon_{is} \, dV = \frac{\bar{u}_{i,sk} + \bar{u}_{s,ik}}{2}$$

$$< \varepsilon^m_{is} >_k = < \varepsilon^b_{is} >_k \quad \forall\, i, s, k = 1, 2, 3 \quad \Rightarrow \quad \bar{u}^m_{i,sk} = \bar{u}^b_{i,sk} = \bar{u}_{i,sk} \quad \forall\, i, s, k = 1, 2, 3$$

So we impose the same displacement field

The energy contribution of the small inclusion is higher order infinitesimal so can be neglected





Then we compare the elastic energies

$$\xi^{he}\left(\mu_m, k_m, \cancel{\mu_i}, \cancel{k_i}\right) - \xi^{ho}\left(\bar{\mu}, \bar{k}\right) \neq 0$$

We fill the second order energy gap with a second order energy due to the Mindlin model

$$\Delta\xi^{he-ho}\left(\mu_m, k_m, \bar{\mu}, \bar{k}\right) = \xi^{ho}_{2ord}$$

Equilibrium condition – lack of generality

The second order displacement field must satisfy the equilibrium condition

$$\bar{u}_{k,ki} = (2\nu - 1)\bar{u}_{i,kk}$$

on both the heterogeneous and homogeneous material.

$$\bar{u}_{k,ki} = (2\nu^m - 1)\bar{u}_{i,kk}$$
$$\bar{u}_{k,ki} = (2\nu^b - 1)\bar{u}_{i,kk}$$

Those two sets of equations are linearly independent so, when we impose the equilibrium into the heterogeneous (m) material, the equilibrium condition of the homogeneous (b) material is not included. Then we can't have the equilibrium into both materials without loosing generality.

The lack of equilibrium of (b) creates an Upper Bound of the elastic energy, so we underestimate the amount of stifness needed to match the energy

$$\xi_m^{Cauchy}(u^m) = \xi_b^{Cauchy+Mindlin}(u^m) \geq \xi_b^{Cauchy+Mindlin}(\tilde{u}^b)$$

The Second Gradient - Mindlin Model

The Second Gradient Mindlin Model uses the second gradient of the displacement to describe a second order elastic potential.

$$\chi_{ijk} = \bar{u}_{k,ij}$$
$$\chi_{ijk} = \chi_{jik}$$

The isotropic Mindlin model uses the 5 invariants of the second gradient tensor

$$\varphi(\chi) = \frac{1}{2} a_i \, Inv(\chi)_i \quad i = 1, ..., 5$$

$$Inv(\chi)_1 = \chi_{iik} \, \chi_{jjk}$$
$$Inv(\chi)_2 = \chi_{iik} \, \chi_{jkj} = \chi_{iik} \, \chi_{kjj}$$
$$Inv(\chi)_3 = \chi_{iki} \, \chi_{jkj} = \chi_{kii} \, \chi_{jkj} = \chi_{kii} \, \chi_{kjj} = \chi_{iki} \, \chi_{kjj}$$
$$Inv(\chi)_4 = \chi_{ijk} \, \chi_{ijk} = \chi_{jik} \, \chi_{ijk} = \chi_{jik} \, \chi_{jik} = \chi_{ijk} \, \chi_{jik}$$
$$Inv(\chi)_5 = \chi_{ijk} \, \chi_{kji} = \chi_{jik} \, \chi_{kji} = \chi_{jik} \, \chi_{jki} = \chi_{ijk} \, \chi_{kji}$$

Energy matching

If we describe the energy gap in terms of the second gradient tensor components

$$\begin{aligned}
\xi_b^{Mindlin}(u_m) &= \Delta\xi_{m-b}^{Cauchy}(u_m) = \xi_m^{Cauchy}(u_m) - \xi_b^{Cauchy}(u_m) \\
&= \frac{h^5}{12}[\frac{\lambda^m}{2}(\bar{u^m}_{i,ij}\,\bar{u^m}_{k,kj}) + \frac{\mu^m}{2}(\bar{u^m}^2_{i,js} + \bar{u^m}_{i,js}\,\bar{u^m}_{s,ji})] + \\
&- \frac{h^5}{12}[\frac{\lambda^b}{2}(\bar{u^m}_{i,ij}\,\bar{u^m}_{k,kj}) + \frac{\mu^b}{2}(\bar{u^m}^2_{i,js} + \bar{u^m}_{i,js}\,\bar{u^m}_{s,ji})] \\
&= \frac{h^5}{12}[\frac{\lambda^m - \lambda^b}{2}(\bar{u^m}_{i,ij}\,\bar{u^m}_{k,kj}) + \frac{\mu^m - \mu^b}{2}(\bar{u^m}^2_{i,js} + \bar{u^m}_{i,js}\,\bar{u^m}_{s,ji})] \\
&= \frac{h^5}{12}[\frac{\lambda^m - \lambda^b}{2}(\chi^m_{iji}\,\chi^m_{kjk}) + \frac{\mu^m - \mu^b}{2}((\chi^m_{jsi})^2 + \chi^m_{jsi}\,\chi^m_{jis})] \\
&= \frac{h^5}{12}[\frac{\lambda^m - \lambda^b}{2} Inv(\chi^m)_3 + \frac{\mu^m - \mu^b}{2}(Inv(\chi^m)_4 + Inv(\chi^m)_5)]
\end{aligned}$$

And considering the Mindlin energy from the elastic potential

$$\xi_b^{Mindlin}(u_m) = \int_V \varphi(\chi^m)\,dV = h^3\,\varphi(\chi^m)$$

Energy matching

We obtain the perfect matching with the solution:

$$a1 = 0$$
$$a2 = 0$$
$$a3 = \frac{h^2}{12}(\lambda^m - \lambda^b)$$
$$a4 = a5 = \frac{h^2}{12}(\mu^m - \mu^b)$$

Conclusions and further developments

The solution is:

* Aproximate, and the error decreases for very small inclusions

* Gives a lower bound estimaion of the stifness contribution needed from the Mindlin model in order to obtain the first and second order homogeneization

We can improve the solution with:

* Imposing the global equilibrium with the contibution of the couple stress generated by the Mindlin model

Thank you for your attention

Acknowledgements: Support from the grant
PIAP-GA-2009-251475-HYDROFRAC is gratefully acknowledged