Dislocations in Prestressed Metals

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Summary

- We generalize the inclusion and the edge dislocation problems, starting from the solutions given by Eshelby (1957) and Willis (1965)
- These are limited to the case of linear isotropic elasticity
- We extend the solutions to the general case of infinite, homogeneouly prestressed and incompressible elastic plane, introducing an incremental formulation
- The incremental displacement and mean stress fields show singularities, which are treated with the Green's functions
- Our solutions for the inclusion problem can be mathematically manipulated to give the field expressions for the dislocation problem
- Two samples of a circular inclusion and an edge dislocation dipole have been implemented in order to make a comparison with the analytic solutions and to understand the role of the prestress

Constitutive framework

- We refer to an incompressiblle nonlinear elastic material deformed under plane strain condition
- **Constitutive equations** (Biot, 1965) and **incompressibility** constraint:

$$\boxed{\dot{t}_{ij} = \mathbb{K}_{ijkl} v_{l,k} + \dot{p}\delta_{ij} \qquad v_{i,i} = 0} \tag{1}$$

- $\mathbb{K}$ has the major simmetry: $\mathbb{K}_{ijkl} = \mathbb{K}_{klij}$
- **Dimensionless prestress** and **anisotropy parameters**:

$$\xi = \frac{\mu_*}{\mu} \qquad \eta = \frac{p}{\mu} = \frac{\sigma_1 + \sigma_2}{2\mu} \qquad \kappa = \frac{\sigma}{2\mu} = \frac{\sigma_1 - \sigma_2}{2\mu} \tag{2}$$

- We will restrict the analysis to the elliptic regime, which corresponds to

$$\mu > 0 \qquad k^2 < 1 \qquad 2\xi > 1 - \sqrt{1 - k^2} \tag{3}$$

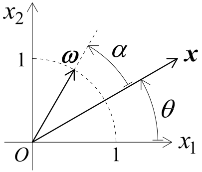
- Introduction of the J_2-deformation theory of plasticity (Hutchinson and Neale, 1979):

$$k = \tanh(2\varepsilon) \qquad \xi = \frac{N\,k}{2\varepsilon} \tag{4}$$

Equilibrium equations and regime classification

- Reference system, vectors $\boldsymbol{\omega}$, $\boldsymbol{x}$ and angles θ, α are shown in Fig. 1
- **Equilibrium equations**:

$$\boxed{\dot{t}_{ij,i} + \dot{f}_j \delta(\boldsymbol{x}) = \rho v_{j,tt}} \qquad (5)$$



- A manipulation of the equilibrium equations gives the ***regime classification***
- Introduction of the operator $L(\boldsymbol{\omega})$ in the characteristic equation:

$$L(\boldsymbol{\omega}) = \mu \omega_2^4 (1+\kappa) \left(\frac{\omega_1^2}{\omega_2^2} - \gamma_1 \right) \left(\frac{\omega_1^2}{\omega_2^2} - \gamma_2 \right) > 0 \quad \text{in E} \qquad (6)$$

- **Plane wave expansion**, with **stream function** $(v_1 = \psi_{,2}\,, v_2 = -\psi_{,1})$ and **Green's tensor** $(v_1^g = \psi^g_{,2}\,, v_2^g = -\psi^g_{,1})$

$$\delta(\boldsymbol{x}) = -\frac{1}{4\pi^2} \oint_{|\boldsymbol{\omega}|=1} \frac{d\omega}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} \qquad \psi^g(\boldsymbol{x}) = -\frac{1}{4\pi^2} \oint_{|\boldsymbol{\omega}|=1} \tilde{\psi}^g(\boldsymbol{\omega}\cdot\boldsymbol{x})\, d\omega \qquad (7)$$

Incremental velocity and mean stress fields

- ***Incremental velocity field***:

$$v_m^g = -\frac{r}{4\pi^2\mu\,(1+\kappa)} \int_0^{2\pi} \sin\left[\alpha+\theta+(1-m)\frac{\pi}{2}\right] \cos\left[\alpha+\theta+ +(2-g)\frac{\pi}{2}\right] \frac{\ln|\cos\alpha|}{\Lambda(\alpha+\theta)}\,d\alpha \qquad (8)$$

- ***Incremental mean stress field***:

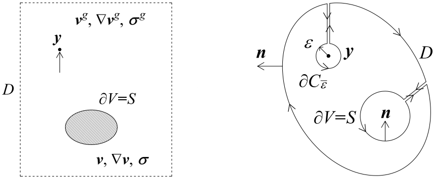
$$\dot{\pi}^1 = -\frac{\cos\theta}{2\pi r} + \frac{1}{4\pi^2(1+k)r} \int_0^{2\pi} \frac{\sin^2(\alpha+\theta)\cos(\alpha+\theta)\Gamma(\alpha+\theta)}{\Lambda(\alpha+\theta)\cos\alpha}\,d\alpha \qquad (9a)$$

$$\dot{\pi}^2 = -\frac{\cos\theta}{2\pi r} - \frac{1}{4\pi^2(1+k)r} \int_0^{2\pi} \frac{\sin(\alpha+\theta)\cos^2(\alpha+\theta)\Gamma(\alpha+\theta)}{\Lambda(\alpha+\theta)\cos\alpha}\,d\alpha \qquad (9b)$$

where:

$$\begin{aligned} \Xi(x) &= \mathrm{Ci}(|x|)\sin x\,\mathrm{Si}(x)\cos x - \frac{\pi}{2}\sin x \\ \Lambda(\alpha) &= \sin^4\alpha\left(\cot^2\alpha-\gamma_1\right)\left(\cot^2\alpha-\gamma_2\right) > 0 \\ \Gamma(\alpha+\theta) &= 2\,(\xi-1)\left[2\cos^2(\alpha+\theta)-1\right]-k \end{aligned} \qquad (10)$$

Geometry and initial conditions



- We consider an infinite region D containing an inclusion of arbitrary shape, with volume V and surface $S = \partial V$ (Fig. 2)
- The inclusion is subject to a prescribed uniform incremental displacement gradient $v^P_{i,j}$ that can be thought as an inelastic (for instance plastic or thermal) deformation
- The inclusion is constrained by the surrounding matrix material, so that an elastic deformation $v^E_{i,j}$ is produced
- The 'total' incremental displacement gradient $v_{i,j}$ can be obtained through the additive rule

$$v_{i,j} = v^E_{i,j} + v^P_{i,j} \tag{11}$$

Incremental displacement field

- The elastic part of the incremental deformation produces the incremental nominal stress

$$\dot{t}_{ij} = \mathbb{K}_{ijkl} v_{l,k} - \mathbb{K}_{ijkl} v^P_{l,k} + \dot{p}\, \delta_{ij} - \dot{p}^P \delta_{ij} \tag{12}$$

- Equilibrium equations for an infinite body containing a concentrated unit force

$$\dot{t}^g_{ij,i}\left(\boldsymbol{y}-\boldsymbol{x}\right) + \delta_{gj}\delta\left(\boldsymbol{y}-\boldsymbol{x}\right) = 0 \tag{13}$$

- We consider the closed smooth domain $D_{\text{out}} = D - C_\varepsilon - V$ (Fig. 2) and apply the Betti's identity

$$\boxed{\int_{D_{\text{out}}} \left[\dot{t}^g_{ij,i}\left(\boldsymbol{y}-\boldsymbol{x}\right) v_j(\boldsymbol{x}) - \dot{t}_{ij,i}(\boldsymbol{x}) v^g_j\left(\boldsymbol{y}-\boldsymbol{x}\right)\right] dV_{\boldsymbol{x}} = 0} \tag{14}$$

- Deviator of the incremental displacement gradient: $\tilde{v}_{i,j} = v_{i,j} - \frac{1}{3} v_{k,k}\delta_{i,j}$
- Application of Gauss theorem and the major simmetry of $\mathbb{K}_{ijkl}$ yields the ***integral equation for the incremental displacements outside the inclusion produced by the uniform inelastic field*** $v^P_{l,k}$

$$v_g(\boldsymbol{y}) = \int_S \mathbb{K}_{ijkl} v^P_{l,k} n_i v^g_j\left(\boldsymbol{y}-\boldsymbol{x}\right) dS_{\boldsymbol{x}} - \int_V \dot{p}^g\left(\boldsymbol{y}-\boldsymbol{x}\right) v^P_{k,k}\, dV_{\boldsymbol{x}} \tag{15}$$

Incremental mean stress field

- The incremental equilibrium equations (13) allow us to derive the gradient of $\dot{p}$ in the form

$$\dot{p}_{,k} = -\mathbb{K}_{jklm}\tilde{v}_{m,lj} \tag{16}$$

- A substitution of the second derivative of (15), together with a manipulation of the term $\mathbb{K}_{sirg}\dot{p}^{g}_{,rs}$ (Bigoni-Capuani, 2002) yields the ***integral equation for the incremental mean stress outside the inclusion produced by the uniform inelastic field*** $v^{P}_{l,k}$

$$\begin{aligned}\dot{p}(\boldsymbol{y}) = &-\int_S \mathbb{K}_{jklm} v^{P}_{m,l} \dot{p}^{k}\left(\boldsymbol{y}-\boldsymbol{x}\right) n_j \, dS_{\boldsymbol{x}} - 2\mu^2 \int_V \Big[\big[4\xi(1-2\xi)+ \\ &+ k(1-k-4\xi)\big] v^{1}_{1,11} - k(1+k) v^{2}_{2,11}\Big] v^{P}_{k,k} \, dV_{\boldsymbol{x}}\end{aligned} \tag{17}$$

Alternative solutions

- It is possible to derive expressions alternative, but equivalent to (15) and (17) (Willis, 1965), simply exploiting the equilibrium equations and the Gauss theorem
- ***Incremental displacement field***, equivalent to (15)

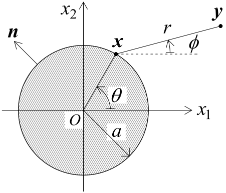
$$v_g(\boldsymbol{y}) = \int_S \left[\mathbb{K}_{jklm} v^g_{k,j}\left(\boldsymbol{y}-\boldsymbol{x}\right) + \dot{p}^g\left(\boldsymbol{y}-\boldsymbol{x}\right)\delta_{lm}\right] v^P_m n_l \, dS_{\boldsymbol{x}} + -2\int_V p^g\left(\boldsymbol{y}-\boldsymbol{x}\right) v^P_{k,k} \, dV_{\boldsymbol{x}} \tag{18}$$

- ***Incremental displacement field***, equivalent to (17)

$$\dot{p}(\boldsymbol{y}) = -\int_S \left[\mathbb{K}_{jklm} \dot{p}^k_{,j}\left(\boldsymbol{y}-\boldsymbol{x}\right) - F\delta_{lm}\right] v^P_m n_l \, dS_{\boldsymbol{x}} - 2\int_V F v^P_{k,k} \, dV_{\boldsymbol{x}} \tag{19}$$

Circular inclusion

- Circular inclusion subject to an inelastic volumetric incremental strain $v^P_{i,j} = \beta\delta_{ij}$



$$
\begin{aligned}
r^2 &= (y_1 - a\cos\theta)^2 + (y_2 - a\sin\theta)^2 \\
\phi &= \arctan\left(\frac{y_2 - a\sin\theta}{y_1 - a\cos\theta}\right)
\end{aligned}
\tag{20}
$$

- Boundary equations for ***incremental displacements***:

$$
v_g(\boldsymbol{y}) = \beta a \int_0^{2\pi} \left[-(k+\eta)n_1 v_1^g + (k-\eta)n_2 v_2^g\right] d\theta - \beta a \int_0^a \int_0^{2\pi} \dot{p}^g \, d\theta \, da \tag{21}
$$

- Boundary equations for ***incremental mean stress***:

$$
\begin{aligned}
\dot{p}(\boldsymbol{y}) = \beta a \int_0^{2\pi} \left[(k+\eta)n_1\dot{p}^1 - (k-\eta)n_2\dot{p}^2\right] d\theta - 2\beta a \int_0^a \int_0^{2\pi} \Big[\Big[4\xi(1+ \\
- 2\xi) + +k(1-k-4\xi)\Big] v^1_{1,11} - k(1+k)v^2_{2,11}\Big] d\theta \, da
\end{aligned}
\tag{22}
$$

Circular inclusion

- Simple case of null prestress ($k = 0$) and isotropic elasticity ($\xi = 1$):

$$v_g = \frac{\beta a}{\pi} \int_0^a \int_0^{2\pi} \frac{y_g - x_g}{r^2} \, d\theta \, da \qquad \dot{p} = 0 \tag{23}$$

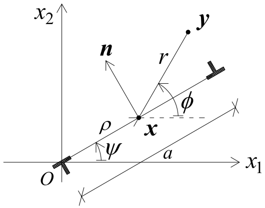
Remarks

- We can obtain the displacement and mean stress fields for the compressible isotropic elastic material ($\nu = 0.5$) simply by changing the constitutive equations in (21) and (22)
- A comparison between these solutions and the solutions of Eshelby can be made, showing the same results
- Our solutions are more general, even in the simple case of compressible isotropic elastic material

Straight edge dislocations dipole

- The integral equations determining the incremental displacement and mean stress can be obtained from equations (18) and (19) by considering a thin (thickness h) rectangular inclusion, (without loss of generality) with one edge centred at the origin and subject to the incremental simple shear displacement field

$$v_i^P = \frac{x_k n_k}{h}\, b_i \qquad b_k n_k = 0 \tag{24}$$



- Taking the limit $h \longrightarrow 0$, we obtain ***the integral equations for a straight edge dislocation in a prestressed material***

$$v_g(\boldsymbol{y}) = \int_D b_m n_l(\boldsymbol{x}) \mathbb{K}_{jklm} v^g_{k,j}\,(\boldsymbol{x} - \boldsymbol{x})\, dD_{\boldsymbol{x}} \tag{25a}$$

$$\dot{p}(\boldsymbol{y}) = -\int_D b_m n_l(\boldsymbol{x}) \mathbb{K}_{jklm} \dot{p}^k_{,j}\,(\boldsymbol{x} - \boldsymbol{x})\, dD_{\boldsymbol{x}} \tag{25b}$$

Straight edge dislocations dipole

- Assuming the reference system shown in Fig. 1 and representing the dislocation line with a polar coordinate system $(\rho,\ \psi)$, where $\rho \in [0, a]$, we have

$$\begin{aligned} \boldsymbol{b} &= b\left\{cos\psi, \sin\psi\right\} & \boldsymbol{n} &= \left\{-\sin\psi, \cos\psi\right\} \\ r^2 &= (y_1 - \rho\cos\psi)^2 + (y_2 - \rho\sin\psi)^2 & \phi &= \arctan\left(\frac{y_2 - \rho\sin\psi}{y_1 - \rho\cos\psi}\right) \end{aligned} \tag{26}$$

- Since $\boldsymbol{b}$ is constant and orthogonal to $\boldsymbol{n}$, the ***incremental displacement*** and ***mean stress fields*** become

$$\begin{aligned} v_g(\boldsymbol{y}) = b\int_0^a \Big[&\Omega_1(\psi)\, v^g_{1,1}(\boldsymbol{y}, \rho, \psi) + \Omega_2(\psi)\, v^g_{1,2}(\boldsymbol{y}, \rho, \psi) + \\ &+ \Omega_3(\psi)\, v^g_{2,1}(\boldsymbol{y}, \rho, \psi)\Big]\, d\rho \end{aligned} \tag{27a}$$

$$\begin{aligned} \dot{p}(\boldsymbol{y}) = -b\int_0^a \Big[&\Omega_2(\psi)\, \dot{p}^1_{,2}(\boldsymbol{y}, \rho, \psi) + \Omega_3(\psi)\, \dot{p}^2_{,1}(\boldsymbol{y}, \rho, \psi) + \\ &+ \Omega_4(\psi)\, \dot{p}^1_{,1}(\boldsymbol{y}, \rho, \psi) + \Omega_5(\psi)\, \dot{p}^2_{,2}(\boldsymbol{y}, \rho, \psi)\Big]\, d\rho \end{aligned} \tag{27b}$$

Straight edge dislocations dipole

where

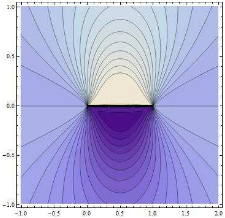
$$
\begin{aligned}
\Omega_1(\psi) &= \mu(\eta - 2\xi)\sin(2\psi) \\
\Omega_2(\psi) &= \mu\left[(1-k)\cos^2\psi - (1-\eta)\sin^2\psi\right] \\
\Omega_3(\psi) &= \mu\left[(1-\eta)\cos^2\psi - (1+k)\sin^2\psi\right] \\
\Omega_4(\psi) &= \frac{\mu}{2}(k+\eta-2\xi)\sin(2\psi) \qquad \Omega_5(\psi) = \frac{\mu}{2}(k-\eta+2\xi)\sin(2\psi)
\end{aligned}
\tag{28}
$$

- In the simple case of null prestress ($k=0$ and $\eta=0$) equations (27) reduce to

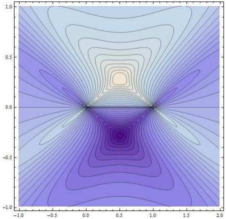
$$
v_g(\boldsymbol{y}) = \mu\, b \int_0^a \Big[-2\xi\, v^g_{1,1}(\boldsymbol{y},\rho,\psi)\sin(2\psi) + \big[v^g_{1,2}(\boldsymbol{y},\rho,\psi) + v^g_{2,1}(\boldsymbol{y},\rho,\psi)\big]\cos(2\psi)\Big]\, d\rho \tag{29a}
$$

$$
\dot{p}(\boldsymbol{y}) = -\mu\, b \int_0^a \Big[\big[\dot{p}^1_{,2}(\boldsymbol{y},\rho,\psi) + \dot{p}^2_{,1}(\boldsymbol{y},\rho,\psi)\big]\cos(2\psi) - \xi\big[\dot{p}^1_{,1}(\boldsymbol{y},\rho,\psi) - \dot{p}^2_{,2}(\boldsymbol{y},\rho,\psi)\big]\sin(2\psi)\Big]\, d\rho \tag{29b}
$$

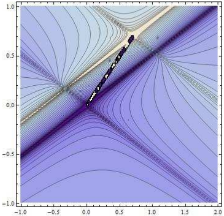
Example: numerical models for $v1$ displacement



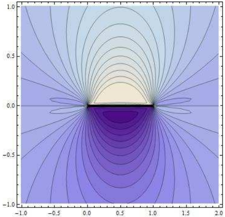
($\psi = 0,\ k = 0.80,\ \xi = 1$)



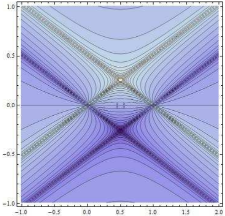
($\psi = 0,\ k = 0.80,\ \xi = 1/4$)



($\psi = \pi/4,\ k = 0.866,\ \xi = 1/4$)

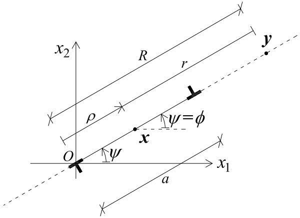


($\psi = 0,\ k = 0.99,\ \xi = 1$)



($\psi = 0,\ k = 0.866,\ \xi = 1/4$)

Displacements and mean stress along the dislocation line

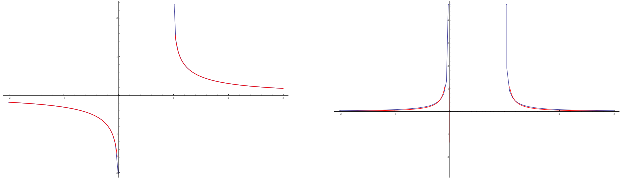


- The displacement and the mean stress fields can be evaluated along the dislocation line through equations (27)
- The point $\boldsymbol{y}$ along the dislocation line is represented by $\boldsymbol{y} = (r+\rho)\{\cos\psi, \sin\psi\}$ and the angle ϕ is constant and equal to ψ
- The Green's function gradient for displacement and mean stress can be expressed as

$$v^g_{i,j} = \frac{1}{r}\bar{v}^g_{i,j}(\alpha,\psi) \qquad \dot{p}^g_{,i} = \frac{1}{r^2}\bar{\dot{p}}^g_{,i}(\alpha,\psi) \tag{30}$$

where $\bar{v}^g_{i,j}$ and $\bar{\dot{p}}^g_{,i}$ are function of the sole variables α and $\phi = \psi$

Displacements and mean stress along the dislocation line



- The dependence on ρ is explicit, so that the ***displacement and mean stress fields along the dislocation*** take the following form

$$v_g(\boldsymbol{y}) = b \ln\left(\frac{R}{R-a}\right)\Big[\Omega_1(\psi)\,\bar{v}^g_{1,1}(\boldsymbol{y},\alpha,\psi) + \Omega_2(\psi)\,\bar{v}^g_{1,2}(\boldsymbol{y},\alpha,\psi) + \Omega_3(\psi)\,\bar{v}^g_{2,1}(\boldsymbol{y},\alpha,\psi)\Big] \tag{31a}$$

$$\dot{p}(\boldsymbol{y}) = -\frac{b\,a}{R(R-a)}\Big[\Omega_2(\psi)\,\dot{\bar{p}}^1_{,2}(\boldsymbol{y},\alpha,\psi) + \Omega_3(\psi)\,\dot{\bar{p}}^2_{,1}(\boldsymbol{y},\alpha,\psi) + \Omega_4(\psi)\,\dot{\bar{p}}^1_{,1}(\boldsymbol{y},\alpha,\psi) + \Omega_5(\psi)\,\dot{\bar{p}}^2_{,2}(\boldsymbol{y},\alpha,\psi)\Big] \tag{31b}$$

- These two equations show a logarithmic and an hyperbolic discontinuity in displacement and mean stress field respectively

State of the art and conclusions

- The inclusion and dislocation problems have been generalized to the case of infinite, homogeneouly prestressed and incompressible elastic plane (incremental formulation)
- The solutions have been also extended to the J_2-flow theory
- A comparison between our solutions (**reduced** to the linear isotropic elastic material) and the classical solutions (**limited** to the linear isotropic elastic material) and shows the perfect equivalence of the results
- Numerical models for the edge dislocation have been implemented in order to investigate the shear band formation near the elliptic border
- Other numerical simulations will be implemented in order to lead to a better understanding of the role of the prestress
- An experiment on the edge dislocation (an innovation in this field) will be made in the next weeks with photoelasticity techniques

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Thank You for Your attention!