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BOUNDARY VALUE PROBLEMS IN SPACES DEFINED BY MODULUS OF CONTINUITY

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Dedicated to the 65th birthday of Professor Antanas Laurinčikas

Abstract. The properties of the singular integral operator in spaces with modulus of continuity are studied. The solution of the Riemann boundary value problems for analytic functions is presented on their base. An effect of the loss of regularity is discussed.

Key words and phrases: boundary value problem, modulus of continuity, singular integral.

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1. Introduction

We consider the Riemann boundary value problem on the unit disk, which consists of determination of a couple of functions $\Psi^+(z)$ and $\Psi^-(z)$, holomorphic on interior set D^+ and exterior set D^- of the unit circle $L = \{z \in \mathbb{C} : |z| = 1\}$, respectively, continuous up to the closure of these domains and satisfying the boundary condition on L:

$$\Psi^+(t) = G(t)\Psi^-(t) + g(t), \quad t \in L, \tag{1.1}$$

where $G(t)$ and $g(t)$ are given continuous on L functions. An attention to this problem is due to its numerous applications (see, e.g., [6], [7]). The classical situation related to the case of Hölder continuous data G and g is widely discussed in the literature. Actually, the Hölder continuity can be generalized via conception of modulus of continuity. Using the same scheme as in [6], we consider the problem when the functions G and g belong to a space defined by modulus of continuity from special classes $\Phi^{\rho}_{A_n}$ and $\Phi^{\rho}_{B_n}$ introduced in [3]. Properties of theses classes as well as functional spaces determined by the corresponding modulus of continuity are discussed in Section 2. The main goal of the paper is to prove the following result (for more extended representation, see Section 4, Theorem 4.1).

PROPOSITION. *Let μ be a modulus of continuity, $\mu \in \Phi^{\rho}_{A_n} \cup \Phi^{\rho}_{B_n}$, $g, G \in C^{\mu\rho^n}(L)$, where $C^{\mu\rho^n}(L)$ is a space defined by μ on L, and $\varkappa$ is the index of the problem. Then the problem* (1.1) *has $\varkappa + 1$ linearly independent solutions $\Psi^{\pm} \in C^{\mu\rho^n}(\overline{D}^{\pm})$ when $\varkappa \geqslant 0$, and a unique solution when $\varkappa = -1$. If $\varkappa < 0$, then the problem has a solution if and only if additional $-\varkappa - 1$ conditions are satisfied.*

The core of the proof is the behaviour of the singular integral with Cauchy kernel on spaces defined by modulus of continuity, which is considered in Section 3. Solution scheme to (1.1) is presented in Section 4 together with a description of the behaviour of solution in terms of the modulus of continuity, showing the main difference with classical approach. Actually, we show that a kind of "logarithmic" effect could appear (cf., [4]).

2. Notation and auxiliary results

2.1. Definition of modulus of continuity

DEFINITION 2.1. *Let μ be a continuous positive function $\mu : (0, l] \to \mathbb{R}$, $\mu(t) \to 0$ when $t \to +0$. It is said that μ is a modulus of continuity if:*

1) *μ is almost increasing, i.e., there exists $c = c(\mu) > 0$ such that, for any $t_1, t_2 \in (0, l]$, $t_1 \leqslant t_2$, the inequality $\mu(t_1) \leqslant c\mu(t_2)$ holds;*
2) *$\varphi(t) = \dfrac{\mu(t)}{t}$ is almost decreasing, i.e., there exists $c_1 = c_1(\mu) > 0$ such that, for any $t_1, t_2 \in (0, l]$, $t_1 \leqslant t_2$, the inequality $\varphi(t_1) \geqslant c_1\varphi(t_2)$ holds.*

Modulus of continuity μ satisfies *the Dini condition* if $\int_0^l \frac{\mu(t)}{t} \mathrm{d}t < \infty$.

The following scales of classes of moduli of continuity were introduced in [3].

DEFINITION 2.2. *Let ρ be a positive continuous almost decreasing function $\rho : (0, l] \to \mathbb{R}$, and $\mu : (0, l] \to \mathbb{R}$ be a modulus of continuity. Fix a number*

$n \in \mathbb{Z}$, $n \geqslant 0$, and assume that $\mu\rho^{n+1}$ is the modulus of continuity, too. It is said that μ belongs to the class $\Phi^{\rho}_{A_n}$ if the following conditions are satisfied:

1_{A_n}) there exists $A > 0$ such that, for any $x \in (0, l]$,

$$\int_0^x \frac{\mu(t)\rho^n(t)}{t}\mathrm{d}t \leqslant A\mu(x)\rho^{n+1}(x);$$

2_{A_n}) there exists $B > 0$ such that, for any $x \in (0, l]$,

$$\int_x^l \frac{\mu(t)}{t^2}\mathrm{d}t \leqslant B\frac{\mu(x)}{x}.$$

Similarly, μ belongs to the class $\Phi^{\rho}_{B_n}$ if the following conditions are satisfied:

1_{B_n}) there exists $A > 0$ such that, for any $x \in (0, l]$,

$$\int_0^x \frac{\mu(t)\rho^{n+1}(t)}{t}\mathrm{d}t \leqslant A\mu(x)\rho^{n+1}(x);$$

2_{B_n}) there exists $B > 0$ such that, for any $x \in (0, l]$,

$$\int_x^l \frac{\mu(t)}{t^2}\mathrm{d}t \leqslant B\frac{\mu(x)\rho(x)}{x}.$$

We recall few known properties of modulus of continuity:
1) $\mu(t)\rho^n(t)$ satisfies the Dini condition for any $\mu \in \Phi^{\rho}_{A_n} \cup \Phi^{\rho}_{B_n}$;
2) $\Phi^{\rho}_{A_{n+1}} \subset \Phi^{\rho}_{A_n}$ and $\Phi^{\rho}_{B_{n+1}} \subset \Phi^{\rho}_{B_n}$ for any $n \in \mathbb{Z}$, $n \geqslant 0$;
3) if ρ is bounded on $(0, l]$ then $\Phi^{\rho}_{A_n} = \Phi^{\rho}_{B_n}$, and coincides with the class Φ, considered, e.g., in [1], [2], [5].

2.2. Spaces defined by modulus of continuity

Let E be an arbitrary connected set $E \subset \overline{\mathbb{C}}$, $E \neq \emptyset$. We denote by $B(E)$ the set of functions $f : E \to \mathbb{C}$ bounded on E.

Definition 2.3. *Let $\mu : (0, l] \to \mathbb{R}$ be a modulus of continuity. A function $f \in B(E)$ is said to belong to the class $C^{\mu}(E)$ if f satisfies the following condition:*

(*) there exists $C > 0$ such that, for any points $z_1, z_2 \in E$, $z_1 \neq \infty$, $z_2 \neq \infty$, $z_1 \neq z_2$, $|z_1 - z_2| \leqslant l$, the inequality $|f(z_1) - f(z_2)| \leqslant C\mu(|z_1 - z_2|)$ is satisfied.

If $z = \infty \in E$, we have to add an extra condition with the same constant C :

(**) for any point $z \in E$, $z \neq \infty$, $|z| \geqslant \frac{1}{l}$, the inequality $|f(\infty) - f(z)| \leqslant C\mu\big(\frac{1}{|z|}\big)$ is satisfied.

We refer to $C^{\mu}(E)$ as to the space defined by the modulus of continuity μ, i.e.,
1. $C^{\mu}(E)$ is a vector subspace of the space of continuous functions $C(E)$.
2. If $\mu_0(t) = t$, then, for any modulus of continuity, μ $C^{\mu_0}(E) \subset C^{\mu}(E)$.

Corollary 2.1. *Let $f \in C^{\mu}(E)$, $M = f(E)$ and $g \in C^{\mu_0}(M)$. Then $(g \circ f) \in C^{\mu}(E)$, where $\circ$ denotes a superposition.*

Lemma 2.1. *Let f_1, $f_2 \in C^{\mu}(E)$. Then*

1) $(f_1 f_2) \in C^{\mu}(E)$;

2) if there exists $m > 0$, $|f_2(z)| > m$ for any $z \in E$, then $\left(\frac{f_1}{f_2}\right) \in C^{\mu}(E)$.

Proof. The proof of the lemma is standard. Its scheme is similar to that in the case of Hölder continuous functions (see, e.g., [6]). □

2.3. Behaviour of functions in the unit disk

We remind here the following result from [3], which in use in what follows.

Lemma 2.2. *Let $G = \{z \in \mathbb{C} : |z| < 1\}$, and f be a holomorphic function in G and continuous in the closure $\overline{G}$. Let μ be a modulus of continuity, $\mu \in \Phi^{\rho}_{A_n} \cup \Phi^{\rho}_{B_n}$. If the following inequality holds for any θ_1, $\theta_2 \in [0, 2\pi)$*

$$|f(\mathrm{e}^{i\theta_1}) - f(\mathrm{e}^{i\theta_2})| \leqslant A\mu(|\mathrm{e}^{i\theta_1} - \mathrm{e}^{i\theta_2}|)\rho^{n}(|\mathrm{e}^{i\theta_1} - \mathrm{e}^{i\theta_2}|)$$

with $A = A(\mu, f)$, then there exists $C = C(\mu, f)$, such that, for all $\zeta_1, \zeta_2 \in \overline{G}$,

$$|f(\zeta_1) - f(\zeta_2)| \leqslant C\mu(|\zeta_1 - \zeta_2|)\rho^{n+1}(|\zeta_1 - \zeta_2|).$$

3. Singular integral with Cauchy kernel in spaces defined by modulus of continuity

Let L be the unit circle $L = \{z \in \mathbb{C} : |z| = 1\}$. It splits an extended complex plane into interior domain $D^{+} = \{z \in \mathbb{C} : |z| < 1\}$ and exterior domain $D^{-} = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$. It is well known (e.g. [6, p. 17]) that the function

$$\Psi(z) = \frac{1}{2\pi i}\int_{L}\frac{\varphi(\tau)}{\tau - z}\mathrm{d}\tau \tag{3.1}$$

is correctly defined on D^+ and D^- for any continuous density $\varphi : L \to \mathbb{C}$. Moreover, it is holomorphic in D^+ and D^-. We denote by $\Psi^+(z)$ and $\Psi^-(z)$ the restriction of $\Psi(z)$ onto D^+ and D^-, respectively. For $z = t \in L$, formula (3.1) determines a singular integral. Its existence is usually understood in the sense of the Cauchy principal value (as, e.g., in the case of Hölder continuous density).

3.1. Existence of the singular integral

Let us consider the contour singular integral

$$\int_L \frac{\varphi(\tau)}{\tau - t} \mathrm{d}\tau, \quad t \in L. \tag{3.2}$$

The Cauchy principal value of (3.2) at any point $t = \mathrm{e}^{i\alpha} \in L$ is the limit

$$\lim_{\theta \to 0} \int_{L \setminus l_\theta} \frac{\varphi(\tau)}{\tau - t} \mathrm{d}\tau,$$

where $l_\theta = (\mathrm{e}^{i(\alpha-\theta)}, \mathrm{e}^{i(\alpha+\theta)})$.

It is known that $\int_L \frac{d\tau}{\tau - t} = i\pi$ for any $t \in L$. Hence, formally

$$\int_L \frac{\varphi(\tau)}{\tau - t} \mathrm{d}\tau = \int_L \frac{\varphi(\tau) - \varphi(t)}{\tau - t} \mathrm{d}\tau + \varphi(t) \int_L \frac{\mathrm{d}\tau}{\tau - t},$$

and it follows that the existence of (3.2) is equivalent to the existence of

$$\psi(t) = \int_L \frac{\varphi(\tau) - \varphi(t)}{\tau - t} \mathrm{d}\tau. \tag{3.3}$$

Lemma 3.1. *Let $\varphi \in C^\mu(L)$ with μ satisfying the Dini condition. Then integral* (3.2) *exists in the sense of the Cauchy principal value.*

Proof. Let $t = \mathrm{e}^{i\alpha}$, $\tau = \mathrm{e}^{i\sigma}$, where $\alpha, \sigma \in [0, 2\pi)$. Then, for any $\theta \in (0, \pi)$,

$$\begin{aligned} \int_{L \setminus l_\theta} \frac{\varphi(\tau) - \varphi(t)}{\tau - t} \mathrm{d}\tau \quad &= \quad \int_{\alpha+\theta}^{\alpha+\pi} \frac{\varphi(\mathrm{e}^{i\sigma}) - \varphi(\mathrm{e}^{i\alpha})}{\mathrm{e}^{i\sigma} - \mathrm{e}^{i\alpha}} i\mathrm{e}^{i\sigma} \mathrm{d}\sigma \\ &\quad + \int_{\alpha-\pi}^{\alpha-\theta} \frac{\varphi(\mathrm{e}^{i\sigma}) - \varphi(\mathrm{e}^{i\alpha})}{\mathrm{e}^{i\sigma} - \mathrm{e}^{i\alpha}} i\mathrm{e}^{i\sigma} \mathrm{d}\sigma. \end{aligned}$$

It suffices to proceed only with the first integral:

$$\int_{\alpha+\theta}^{\alpha+\pi} \left| \frac{\varphi(\mathrm{e}^{i\sigma}) - \varphi(\mathrm{e}^{i\alpha})}{\mathrm{e}^{i\sigma} - \mathrm{e}^{i\alpha}} \mathrm{e}^{i\sigma} \right| \mathrm{d}\sigma$$

$$\leqslant C_1 \int_{\alpha+\theta}^{\alpha+\pi} \frac{|\varphi(\mathrm{e}^{i\sigma}) - \varphi(\mathrm{e}^{i\alpha})|}{|\sigma-\alpha|}\mathrm{d}\sigma$$
$$\leqslant C_2 \int_{\alpha+\theta}^{\alpha+\pi} \frac{\mu(|\sigma-\alpha|)}{|\sigma-\alpha|}\mathrm{d}\sigma = C_2 \int_{\theta}^{\pi} \frac{\mu(x)}{x}\mathrm{d}x \leqslant C_3 \int_{0}^{\pi} \frac{\mu(x)}{x}\mathrm{d}x < +\infty.$$

□

COROLLARY 3.1. *Singular integral* (3.2) *exists in the sense of the Cauchy principal value and defines a function on* L *(which we will denote as* $\Psi(t)$*) for any* $\mu \in \Phi^{\rho}_{A_n} \cup \Phi^{\rho}_{B_n}$ *and* $\varphi \in C^{\mu\rho^n}(L)$.

3.2. Behaviour of the singular integral on the contour

LEMMA 3.2. *Let* μ *be a modulus of continuity* $\mu \in \Phi^{\rho}_{A_n} \cup \Phi^{\rho}_{B_n}$ *and* $\varphi \in C^{\mu\rho^n}(L)$. *Then the function* ψ *in* (3.3) *satisfies* $\psi \in C^{\mu\rho^{n+1}}(L)$.

Proof. Let $t_1 = \mathrm{e}^{i\alpha_1}$, $t_2 = \mathrm{e}^{i\alpha_2} \in L$, $|\alpha_1 - \alpha_2| \leqslant \pi$. We have to show that

$$|\psi(t_1) - \psi(t_2)| \leqslant C\mu(|t_1 - t_2|)\rho^{n+1}(|t_1 - t_2|),$$

where the constant C is independent of t_1 and t_2. First, let $|\alpha_1 - \alpha_2| \leqslant \frac{\pi}{3}$. Then

$$\begin{aligned}\psi(t_1) - \psi(t_2) \quad = \quad & \underbrace{\int_{l_\theta} \frac{\varphi(\tau) - \varphi(t_1)}{\tau - t_1}\mathrm{d}\tau}_{I_1} - \underbrace{\int_{l_\theta} \frac{\varphi(\tau) - \varphi(t_2)}{\tau - t_2}\mathrm{d}\tau}_{I_2} \\ & \underbrace{-\int_{L\setminus l_\theta} \frac{\varphi(t_1) - \varphi(t_2)}{\tau - t_1}\mathrm{d}\tau}_{I_3} - \underbrace{\int_{L\setminus l_\theta} \frac{[\varphi(\tau) - \varphi(t_2)][t_2 - t_1]}{(\tau - t_1)(\tau - t_2)}\mathrm{d}\tau}_{I_4}.\end{aligned}$$

We consider only $\mu \in \Phi^{\rho}_{A_n}$. The case $\mu \in \Phi^{\rho}_{B_n}$ is studied similarly. If $\mu \in \Phi^{\rho}_{A_n}$, then we can estimate $|I_1|$ in the following way:

$$\begin{aligned}|I_1| \quad \leqslant \quad & \int_{l_\theta} \frac{|\varphi(\tau) - \varphi(t_1)|}{|\tau - t_1|}|\mathrm{d}\tau| \leqslant A_1 \int_0^{\theta} \frac{\mu(x)\rho^n}{x}\mathrm{d}x \\ \leqslant \quad & A_2\mu(2|t_1 - t_2|)\rho^{n+1}(2|t_1 - t_2|) \\ \leqslant \quad & C_1\mu(|t_1 - t_2|)\rho^{n+1}(|t_1 - t_2|).\end{aligned}$$

Analogously,

$$|I_2| \leqslant C_2\mu(|t_1 - t_2|)\rho^{n+1}(|t_1 - t_2|).$$

For $|I_3|$, we have

$$|I_3| \leqslant |\varphi(t_1) - \varphi(t_2)| \left| \int_{L \setminus l_\theta} \frac{\mathrm{d}\tau}{\tau - t_1} \right| \leqslant C_3 \mu(|t_1 - t_2|) \rho^{n+1}(|t_1 - t_2|).$$

Next, for any $\tau \in L \setminus l_\theta$, we have that

$$|\tau - t_2| \leqslant |t_1 - t_2| + |\tau - t_1| \leqslant \frac{1}{2}|\tau - t_1| + |\tau - t_1| \leqslant 2|\tau - t_1|,$$

implying that

$$\begin{aligned} |I_4| &\leqslant \frac{1}{2}|t_2 - t_1| \int_{L \setminus l_\theta} \frac{|\varphi(\tau) - \varphi(t_2)|}{|\tau - t_2|^2} \mathrm{d}\tau \\ &\leqslant A_3 |t_2 - t_1| \int_\theta^l \frac{\mu(x)\rho^n(x)}{x^2} \mathrm{d}x \leqslant A_4 |t_2 - t_1| \rho^n(\theta) \int_\theta^l \frac{\mu(x)}{x^2} \mathrm{d}x \\ &\leqslant A_5 |t_2 - t_1| \rho^n(\theta) \frac{\mu(\theta)}{\theta} \leqslant C_4 \mu(|t_1 - t_2|) \rho^{n+1}(|t_1 - t_2|). \end{aligned}$$

Thus, for any $\mu \in \Phi^\rho_{A_n} \cup \Phi^\rho_{B_n}$, in the case $|\alpha_1 - \alpha_2| \leqslant \frac{\pi}{3}$, we have

$$|\psi(t_1) - \psi(t_2)| \leqslant C_0 \mu(|t_1 - t_2|) \rho^{n+1}(|t_1 - t_2|).$$

This completes the proof since other cases can be reduced to the considered one. $\square$

Corollary 3.2. *Under conditions of* Lemma 3.2*, we have* $\Psi \in C^{\mu\rho^{n+1}}(L)$.

3.3. Continuity of the singular integral on closed domain

Let us fix a point $t \in L$, then the Cauchy integral theorem yields

$$\begin{aligned} \Psi^+(z) &= \frac{1}{2\pi i} \int_L \frac{\varphi(\tau) - \varphi(t)}{\tau - z} \mathrm{d}\tau + \varphi(t), \quad z \in D^+, \\ \Psi^-(z) &= \frac{1}{2\pi i} \int_L \frac{\varphi(\tau) - \varphi(t)}{\tau - z} \mathrm{d}\tau, \quad z \in D^-. \end{aligned}$$

Let us denote

$$\psi(z) = \int_L \frac{\varphi(\tau) - \varphi(t)}{\tau - z} \mathrm{d}\tau, \quad z \in D^+ \quad \text{or} \quad z \in D^-.$$

Lemma 3.3. *Let μ be a modulus of continuity, $\mu \in \Phi^\rho_{A_n} \cup \Phi^\rho_{B_n}$, and $\varphi \in C^{\mu\rho^n}(L)$. Then the functions Ψ^+ and Ψ^- have limit values at any point $t \in L$.*

Proof. We prove the lemma only for Ψ^+, since the proof for Ψ^- is similar. First, we show that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\psi(r\mathrm{e}^{i\alpha}) - \psi(r\mathrm{e}^{i\alpha})| < \varepsilon$ for any $\alpha \in [0, 2\pi)$, whenever $1 - r < \delta$. Let $\alpha \in [0, 2\pi)$ be fixed and $l_\theta = (\mathrm{e}^{i(\alpha-\theta)}, \mathrm{e}^{i(\alpha+\theta)})$ for some $\theta \in (0, \pi)$. Then

$$\begin{aligned}
&\psi(r\mathrm{e}^{i\alpha}) - \psi(\mathrm{e}^{i\alpha}) \\
&\quad = \int_L \frac{r\mathrm{e}^{i\alpha} - \mathrm{e}^{i\alpha}}{\tau - r\mathrm{e}^{i\alpha}} \cdot \frac{\varphi(\tau) - \varphi(t)}{\tau - t}\mathrm{d}\tau \\
&\quad = \underbrace{\int_{l_\theta} \frac{r\mathrm{e}^{i\alpha} - \mathrm{e}^{i\alpha}}{\tau - r\mathrm{e}^{i\alpha}} \cdot \frac{\varphi(\tau) - \varphi(t)}{\tau - t}\mathrm{d}\tau}_{I_1} + \underbrace{\int_{L\setminus l_\theta} \frac{r\mathrm{e}^{i\alpha} - \mathrm{e}^{i\alpha}}{\tau - r\mathrm{e}^{i\alpha}} \cdot \frac{\varphi(\tau) - \varphi(t)}{\tau - t}\mathrm{d}\tau}_{I_2}.
\end{aligned}$$

Let $t = \mathrm{e}^{i\alpha}$, $z = r\mathrm{e}^{i\alpha}$. We consider the triangle $zt\tau$ for any point $\tau = \mathrm{e}^{i(\alpha+\sigma)} \in l_\theta$, denoting $\beta = \angle zt\tau$ and $\gamma = \angle z\tau t$. Then

$$\frac{|z - t|}{|\tau - z|} = \frac{\sin\gamma}{\sin\beta} = \frac{\sin\gamma}{\cos\frac{\sigma}{2}} \leqslant \frac{1}{\cos(\theta/2)}.$$

Thus,

$$\begin{aligned}
|I_1| &\leqslant \frac{1}{\cos(\theta/2)} \int_{l_\theta} \frac{|\varphi(\tau) - \varphi(t)|}{|\tau - t|}|\mathrm{d}\tau| \leqslant \frac{C}{\cos(\theta/2)} \int_0^\theta \frac{\mu(x)\rho^n(x)}{x}\mathrm{d}x \\
&\leqslant C_1\mu(\theta)\rho^{n+1}(\theta).
\end{aligned}$$

Hence, for any $\varepsilon > 0$, we can find $\theta \in (0, \pi)$ such that $|I_1| < \varepsilon$ uniformly in $\alpha \in [0, 2\pi)$. Let us fix such a number θ. The function $\frac{\varphi(\tau)-\varphi(t)}{\tau-t}$ is continuous on $L \setminus l_\theta$, hence, bounded on $L \setminus l_\theta$ by certain $M > 0$. Besides, for any $\tau \in L \setminus l_\theta$, it follows that

$$|\tau - z| \geqslant |\mathrm{e}^{i(\alpha-\theta)} - r\mathrm{e}^{i\alpha}| \geqslant \sin\theta.$$

So, we have the following estimation for $|I_2|$:

$$|I_2| \leqslant \int_{L\setminus l_\theta} \frac{|z - t|}{|\tau - z|} \left|\frac{\varphi(\tau) - \varphi(t)}{\tau - t}\right| |\mathrm{d}\tau| \leqslant \frac{M}{\sin\theta} 2\pi|1 - r|.$$

For the same ε, we can find $\delta > 0$ such that $|I_2| < \varepsilon$ for all $\alpha \in [0, 2\pi)$, whenever $1 - r < \delta$. This gives us the desired inequality $|\psi(r\mathrm{e}^{i\alpha}) - \psi(r\mathrm{e}^{i\alpha})| < \varepsilon$ for all $\alpha \in [0, 2\pi)$.

Now let $z = r\mathrm{e}^{i\alpha} \to t = \mathrm{e}^{i\alpha_0}$. Then $r \to 1$ and $\alpha \to \alpha_0$. Since

$$|\psi(z) - \psi(t)| \leqslant |\psi(r\mathrm{e}^{i\alpha}) - \psi(\mathrm{e}^{i\alpha})| + |\psi(\mathrm{e}^{i\alpha}) - \psi(\mathrm{e}^{i\alpha_0})|,$$

the final result follows from the continuity of $\psi(z)$ (shown above), and the continuity of $\psi(t)$ on L (see the corollary of Lemma 3.2). $\square$

COROLLARY 3.3. *If μ is a modulus of continuity, $\mu \in \Phi^{\rho}_{A_n} \cup \Phi^{\rho}_{B_n}$, and $\varphi \in C^{\mu\rho^n}(L)$, then the* Sokhotski–Plemelj formulas *hold:*

$$\Psi^{+}(t) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}\mathrm{d}\tau + \frac{1}{2}\varphi(t), \tag{3.4}$$

$$\Psi^{-}(t) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}\mathrm{d}\tau - \frac{1}{2}\varphi(t), \tag{3.5}$$

where $\Psi^{\pm}(t)$ denote the limit values on L of the functions $\Psi^{\pm}(z)$, respectively.

3.4. Behaviour of the integral with Cauchy kernel inside the unit disk

We present the behaviour of the functions $\Psi^{+}(z)$ and $\Psi^{-}(z)$ in the form of corollaries from Lemmas 2.2 and 3.2.

COROLLARY 3.4. *If the modulus of continuity $\mu \in \Phi^{\rho}_{A_{n+1}} \cup \Phi^{\rho}_{B_{n+1}}$ and the function $\varphi \in C^{\mu\rho^n}(L)$, then $\Psi^{+}(z) \in C^{\mu\rho^{n+2}}(\overline{D^{+}})$.*

Using reflection with respect to the unit circle, we obtain a similar result for the outer domain D^{-}, namely, we have the statement

COROLLARY 3.5. *If the modulus of continuity $\mu \in \Phi^{\rho}_{A_{n+1}} \cup \Phi^{\rho}_{B_{n+1}}$ and the function $\varphi \in C^{\mu\rho^n}(L)$, then $\Psi^{-}(z) \in C^{\mu\rho^{n+2}}(\overline{D^{-}})$.*

4. Riemann boundary value problem in spaces defined by modulus of continuity

In this section, we study the Riemann boundary value problem (1.1) in spaces defined by modulus of continuity following the scheme of [6, pp. 106–111].

4.1. Jump problem

First, we consider a special case of (1.1), called the jump problem, that is,

$$\Psi^{+}(t) - \Psi^{-}(t) = g(t), \quad t \in L. \tag{4.1}$$

If $\mu \in \Phi^{\rho}_{A_{n+1}} \cup \Phi^{\rho}_{B_{n+1}}$ and $g \in C^{\mu\rho^n}(L)$, then from Sokhotski–Plemelj formulas (3.4) and (3.5), it follows that the unique solution to the jump problem (4.1) has the form of the Cauchy type integral (3.1) with density $\varphi(t) = g(t)$. It follows from the properties of $\Psi(z)$, shown in the previous section, that $\Psi^{+} \in C^{\mu\rho^{n+2}}(\overline{D^{+}})$ and $\Psi^{-} \in C^{\mu\rho^{n+2}}(\overline{D^{-}})$.

4.2. Homogeneous problem

Let us consider the homogeneous Riemann boundary value problem, that is,

$$\Psi^+(t) = G(t)\Psi^-(t), \quad t \in L. \tag{4.2}$$

Solution of the problem depends on the number $\varkappa = \mathrm{Ind} G = \frac{1}{2\pi}[\arg G(t)]_L$, the so-called index of the function G. As in [6, p. 109], we introduce so-called canonical function

$$X^+(z) = \mathrm{e}^{\Gamma^+(z)}, \quad z \in D^+, \quad \text{and} \quad X^-(z) = \frac{\mathrm{e}^{\Gamma^-(z)}}{z^{\varkappa}}, \quad z \in D^-, \tag{4.3}$$

where

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\log(G(\tau)/\tau^{\varkappa})}{\tau - z}\mathrm{d}\tau.$$

Then:

1) $X^+ \in H(D^+)$, $X^- \in H(D^-)$, and $X^+(z)$, $X^-(z)$ can be continuously extended up to L. Moreover, $X^+ \in C^{\mu\rho^{n+2}}(\overline{D}^+)$ and $X^- \in C^{\mu\rho^{n+2}}(\overline{D}^-)$;
2) the functions X^+ and X^- satisfy (4.2).

Lemma 4.1. *Let $\mu \in \Phi^\rho_{A_{n+1}} \cup \Phi^\rho_{B_{n+1}}$, $G \in C^{\mu\rho^n}(L)$ and $\varkappa \in \mathbb{Z}$.*

1. If $\varkappa < 0$, then the problem does not have any other solutions besides trivial.

2. If $\varkappa \geqslant 0$, then the problem has $\varkappa + 1$ linearly independent solutions

$$\Psi_k^+(z) = z^k \mathrm{e}^{\Gamma^+(z)}, \quad z \in \overline{D}^+, \quad \Psi_k^-(z) = z^{k-\varkappa}\mathrm{e}^{\Gamma^-(z)}, \quad z \in \overline{D}^-,$$

$k = 0, 1, \ldots, \varkappa$, Γ is defined by formula (4.3). Also, all functions $\Psi_k^+ \in C^{\mu\rho^{n+2}}(\overline{D}^+)$, $\Psi_k^- \in C^{\mu\rho^{n+2}}(\overline{D}^-)$.

Proof. The proof is straightforward, and repeats the proof in the case of Hölder-continuous data (see [6]). □

4.3. General case

Theorem 4.1. *Let us consider the Riemann boundary problem* (1.1)

$$\Psi^+(t) = G(t)\Psi^-(t) + g(t), \quad t \in L.$$

Let $\mu \in \Phi^\rho_{A_{n+3}} \cup \Phi^\rho_{B_{n+3}}$, and $g, G \in C^{\mu\rho^n}(L)$, $G(t) \neq 0$. Then:

1) if $\varkappa \geqslant 0$, then the solution of the problem can be represented in the form

$$\Psi(z) = X(z)(\Theta(z) + P_{\varkappa}(z)), \tag{4.4}$$

where $X(z)$ is the canonical function, $P_{\varkappa}(z)$ is a polynomial of order $\varkappa$ with arbitrary coefficients, and

$$\Theta(z) = \frac{1}{2\pi i}\int_L \frac{g(\tau)}{X^+(\tau)} \cdot \frac{\mathrm{d}\tau}{\tau - z}; \tag{4.5}$$

2) if $\varkappa = -1$, then the problem has the unique solution represented by formula (4.4) with $P_{\varkappa}(z) \equiv 0$;

3) if $\varkappa < -1$, then there exists the unique solution to the problem if and only if

$$\int_L \frac{g(\tau)}{X^+(\tau)}\tau^{k-1}\mathrm{d}\tau = 0, \quad k = 0, 1, \ldots, -\varkappa - 1;$$

this solution is given by formula (4.4) if we put $P_{\varkappa}(z) \equiv 0$.

In any case, $\Psi^+ \in C^{\mu\rho^{n+4}}(\overline{D}^+)$ and $\Psi^- \in C^{\mu\rho^{n+4}}(\overline{D}^-)$.

Proof. The procedure to obtain a solution is the same as for Hölder continuous data. We are interesting here in the behaviour of the solutions. We can rewrite (1.1) as

$$\frac{\Psi^+(t)}{X^+(t)} = \frac{\Psi^-(t)}{X^-(t)} + \frac{g(t)}{X^+(t)}, \quad t \in L.$$

Then we consider the jump problem with the function $g_1(t) = \frac{g(t)}{X^+(t)}$, $t \in L$. Therefore,

$$g_1(t) = \frac{g(t)}{X^+(t)} = \Theta^+(t) - \Theta^-(t), \quad t \in L,$$

where $\Theta(z)$ is defined by (4.5). It follows that $g_1 \in C^{\mu\rho^{n+2}}(L)$. Thus, $\Theta^+ \in C^{\mu\rho^{n+4}}(\overline{D}^+)$ and $\Theta^- \in C^{\mu\rho^{n+4}}(\overline{D}^-)$. As it can be seen in [6, pp. 111–112], further arguments do not impact on the behaviour of the solution. □

REMARK 4.1. The behaviour of the Cauchy type integral (3.1) impacts on the behaviour of the solution to the problem (1.1): we obtain a *loss of regularity* effect if a given function ρ is unbounded on $(0, l]$.

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